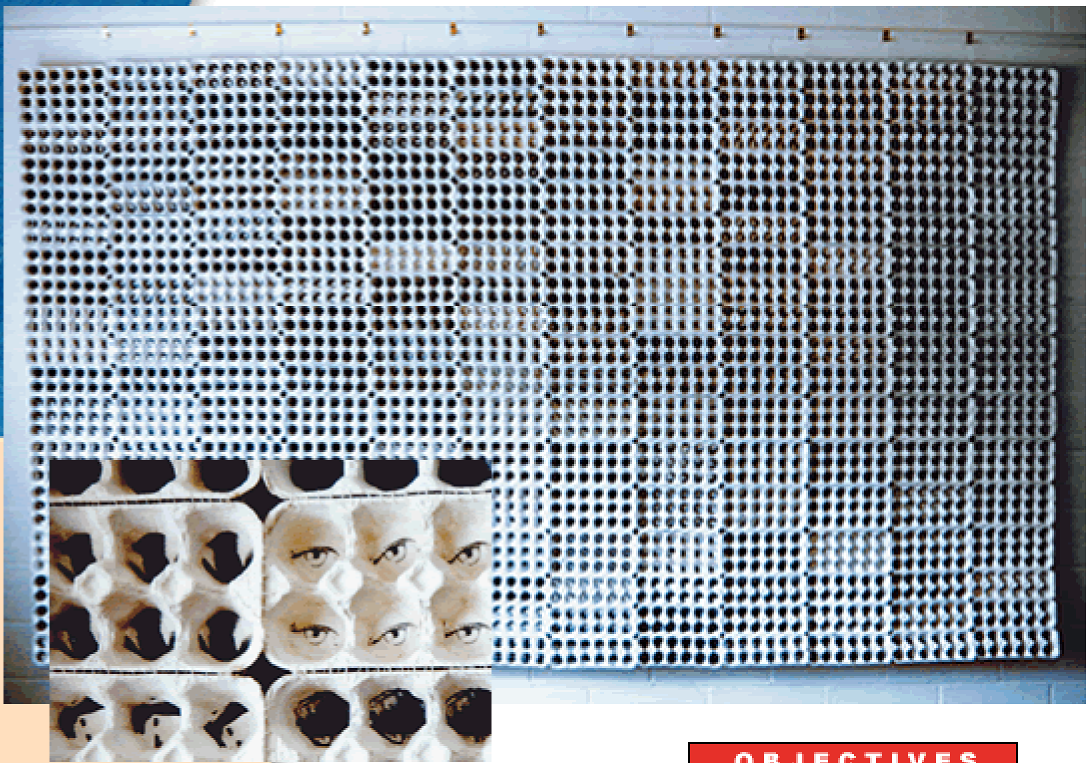
CHAPTER

6

Matrices and Linear Systems



American installation artist Amy Stacey Curtis (b 1970) created this sculpture. The rectangular arrangement of egg cartons is used to organize an even larger arrangement of photocopied images. The egg cartons and their compartments divide the piece into rows and columns, while the small images—some darker or lighter than others—help certain elements of the piece to stand out more prominently.

Fragile and detail from *Fragile* by Amy Stacey Curtis Egg cartons, acrylic, dye, thread, beads, photocopies

OBJECTIVES

In this chapter you will

- **use matrices to organize information**
- **add, subtract, and multiply matrices**
- **solve systems of linear equations with matrices**
- **graph two-variable inequalities on a coordinate plane and solve**
- **write and graph inequalities that represent conditions that must be met simultaneously**

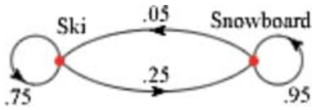
LESSON 6.1

Matrix Representations

All dimensions are critical dimensions, otherwise why are they there?

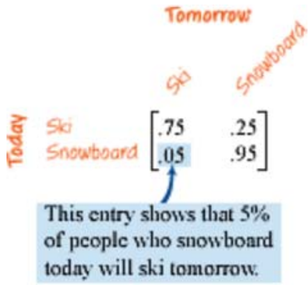
RUSS ZANDBERGEN

On Saturday, Karina surveyed visitors to Snow Mountain with weekend passes and found that 75% of skiers planned to ski again the next day and 25% planned to snowboard. Of the snowboarders, 95% planned to snowboard the next day and 5% planned to ski. In order to display the information, she made this diagram.



The arrows and labels show the patterns of the visitors' next-day activities. For instance, the circular arrow labeled .75 indicates that 75% of the visitors skiing one day plan to ski again the next day. The arrow labeled .25 indicates that 25% of the visitors who ski one day plan to snowboard the next day.

Diagrams like these are called **transition diagrams** because they show how something changes from one time to the next. The same information is sometimes represented in a **transition matrix.** A **matrix** is a rectangular arrangement of numbers. For the Snow Mountain information, the transition matrix looks like this:

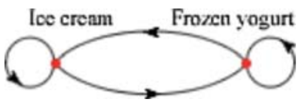


In the investigation you will create a transition diagram and matrix for another situation. You will also use the information to determine how the numbers of people in two different categories change over a period of time.

Investigation
Chilly Choices

The school cafeteria offers a choice of ice cream or frozen yogurt for dessert once a week. During the first week of school, 220 students choose ice cream but only 20 choose frozen yogurt. During each of the following weeks, 10% of the frozen-yogurt eaters switch to ice cream and 5% of the ice-cream eaters switch to frozen yogurt.

Step 1 Complete a transition diagram that displays this information.



Step 2 Complete a transition matrix that represents this information. The rows should indicate the present condition, and the columns should indicate the next condition after the transition.

Step 3 In the second week, how many students choose ice cream and how many students choose frozen yogurt?

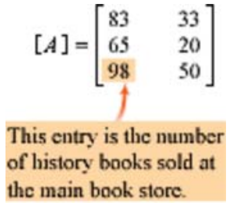
Step 4 How many will choose each option in the third week?

Step 5 Write a recursive routine to take any week's values and give the next week's values.

Step 6 What do you think will happen to the long-run values of the number of students who choose ice cream and the number who choose frozen yogurt?

You can use matrices to organize many kinds of information. For example, the matrix below can be used to represent the number of math, science, and history textbooks sold this week at the main and branch campus bookstores. The rows represent math, science, and history, from top to bottom, and the columns represent the main and branch bookstores, from left to right.

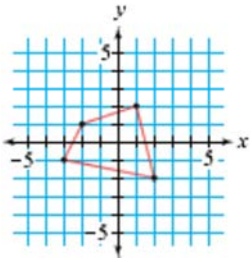
The **dimensions** of the matrix give the numbers of rows and columns, in this case, 3 × 2 (read "three by two"). Each number in the matrix is called an **entry,** or element, and is identified as a_{ij} where i and j are the row number and column number, respectively. In matrix $[A]$ at right, $a_{21} = 65$ because 65 is the entry in row 2, column 1.



Example A shows how to use matrices to represent coordinates of geometric figures.

EXAMPLE A

Represent quadrilateral *ABCD* as a matrix, $[M]$.



▶ Solution

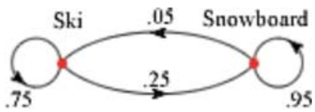
You can use a matrix to organize the coordinates of the consecutive vertices of a geometric figure. Because each vertex has 2 coordinates and there are 4 vertices, use a 2 × 4 matrix with each column containing the x- and y-coordinates of a vertex. Row 1 contains consecutive x-coordinates and row 2 contains the corresponding y-coordinates.

$$[M] = \begin{bmatrix} 1 & -2 & -3 & 2 \\ 2 & 1 & -1 & -2 \end{bmatrix}$$

Example B shows how a transition matrix can be used to organize data and predictions. In Lesson 6.2, you'll learn how to do computations with matrices.

EXAMPLE B

In Karina's survey from the beginning of this lesson, she interviewed 260 skiers and 40 snowboarders. How many people will do each activity the next day if her transition predictions are correct?



$$\begin{bmatrix} .75 & .25 \\ .05 & .95 \end{bmatrix}$$

▶ Solution

The next day, 75% of the 260 skiers will ski again and 5% of the 40 snowboarders will switch to skiing.

Skiers: 260(.75) + 40(.05) = 197

So, 197 people will ski the next day.

The next day, 25% of the 260 skiers will switch to snowboarding and 95% of the 40 snowboarders will snowboard again.

Snowboarders: 260(.25) + 40(.95) = 103

So, 103 people will snowboard the next day.

You can organize the information for the first day and second day as matrices in the form [*number of skiers*   *number of snowboarders*].

[260   40]
[197   103]

You can use transition diagrams and transition matrices to show changes in a closed system. (A closed system is one in which items may change, but nothing is added or removed.) The diagram, though very informative for simple problems, is difficult to use when you have 5 or more starting conditions, as this would create 25 or more arrows, or paths. The transition matrix is just as easy to read for any number of starting conditions as it is for two. It grows in size, but each entry shows what percentage changes from one condition to another.

EXERCISES

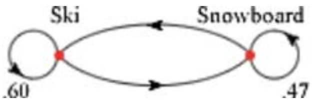
Practice Your Skills

1. Russell collected data similar to Karina's at Powder Hill Resort. He found that 86% of the skiers planned to ski the next day and 92% of the snowboarders planned to snowboard the next day.

a. Draw a transition diagram for Russell's information.

b. Write a transition matrix for the same information. Remember that rows indicate the present condition and columns indicate the next condition. List skiers first and snowboarders second.

2. Complete this transition diagram:



3. Write a transition matrix for the diagram in Exercise 2. Order your information as in Exercise 1b.

4. Matrix [*M*] represents the vertices of △*ABC*.

$$[M] = \begin{bmatrix} -3 & 1 & 2 \\ 2 & 3 & -2 \end{bmatrix}$$

a. Name the coordinates of the vertices and draw the triangle.

b. What matrix represents the image of △*ABC* after a translation down 4 units?

c. What matrix represents the image of △*ABC* after a translation right 4 units?

5. During a recent softball tournament, information about which side players bat from was recorded in a matrix. Row 1 represents girls and row 2 represents boys. Column 1 represents left-handed batters, column 2 represents right-handed batters, and column 3 represents those who can bat with either hand.

$$[A] = \begin{bmatrix} 5 & 13 & 2 \\ 4 & 18 & 3 \end{bmatrix}$$

a. How many girls and how many boys participated in the tournament?
b. How many boys batted right-handed?
c. What is the meaning of the value of a_{12}?

Reason and Apply

6. A mixture of 40 mL of NO and 200 mL of N_2O_2 is heated. During each second at this new temperature, 10% of the NO changes to N_2O_2 and 5% of the N_2O_2 changes to NO.

This photo shows red blood cells, some deformed by sickle cell anemia. Researchers have found that nitric oxide (NO) counteracts the effects of sickle cell anemia.

a. Draw a transition diagram that displays this information.
b. Write a transition matrix that represents the same information. List NO first and N_2O_2 second.
c. If the total amount remains at 240 mL and the transition percentages stay the same, what are the amounts in milliliters of NO and N_2O_2 after 1 s? After 2 s? Write your answers as matrices in the form $[\text{NO} \quad N_2O_2]$.

7. In many countries, more people move into the cities than out of the cities. Suppose that in a certain country, 10% of the rural population moves to the city each year and 1% of the urban population moves out of the city each year.

a. Draw a transition diagram that displays this information.
b. Write a transition matrix that represents this same information. List urban dwellers first and rural dwellers second.
c. If 16 million of the country's 25 million people live in the city initially, what are the urban and rural populations in millions after 1 yr? After 2 yr? Write your answers as matrices in the form $[urban \quad rural]$.

8. Recall the matrix [*A*] on page 302 that represents the number of math, science, and history textbooks sold at the main and branch campus bookstores this week.

$$[A] = \begin{bmatrix} 83 & 33 \\ 65 & 20 \\ 98 & 50 \end{bmatrix}$$

a. Explain the meaning of the value of a_{32}.
b. Explain the meaning of the value of a_{21}.
c. Matrix [*B*] represents last week's sales. Compare this week's sales of math books with last week's sales.

$$[B] = \begin{bmatrix} 80 & 25 \\ 65 & 15 \\ 105 & 55 \end{bmatrix}$$

d. Write a matrix that represents the total sales during last week and this week.

9. The three largest categories of motor vehicles are sedan, SUV, and minivan. Suppose that of the buyers in a particular community who now own a minivan, 18% will change to an SUV and 20% will change to a sedan. Of the buyers who now own a sedan, 35% will change to a minivan and 20% will change to an SUV, and of those who now own an SUV, 12% will buy a minivan and 32% will buy a sedan.

a. Draw a transition diagram that displays these changes.
b. Write a transition matrix that represents this scenario. List the rows and columns in the order minivan, sedan, SUV.
c. What is the sum of the entries in row 1? Row 2? Row 3? Why does this sum make sense?

10. Lisa Crawford is getting into the moving-truck rental business in three nearby counties. She has the funds to buy about 100 trucks. Her studies show that 20% of the trucks rented in Bay County go to Sage County, and 15% go to Thyme County. The rest start and end in Bay County. From Sage County, 25% of rentals go to Bay and 55% stay in Sage, whereas the rest move to Thyme County. From Thyme County, 40% of rentals end in Bay and 30% in Sage.

a. Draw a transition diagram that displays this information.

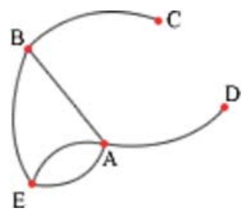
b. Write a transition matrix that represents this scenario. List your rows and columns in the order Bay, Sage, Thyme.

c. What is the sum of the entries in row 1? Row 2? Row 3? Why does this sum make sense?

d. If Lisa starts with 45 trucks in Bay County, 30 trucks in Sage County, and 25 trucks in Thyme County, and all trucks are rented one Saturday, how many trucks will she expect to be in each county the next morning?

11. **APPLICATION** Fly-Right Airways operates routes out of five cities as shown in the route map below. Each segment connecting two cities represents a round-trip flight between them. Matrix $[M]$ displays the information from the map in matrix form with the cities, A, B, C, D, and E, listed in order in the rows and columns. The rows represent starting conditions (departure cities), and the columns represent next conditions (arrival cities). This matrix is called an adjacency matrix. For instance, the value of the entry in row 1, column 5 shows that there are two round-trip flights between City A and City E.

$$[M] = \begin{bmatrix} 0 & 1 & 0 & 1 & 2 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \end{bmatrix}$$

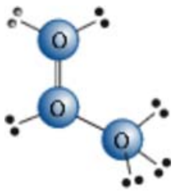


a. What are the dimensions of this matrix?
b. What is the value of m_{32}? What does this entry represent?
c. Which city has the most flights? Explain how you can tell using the route map and using the matrix.
d. Matrix $[N]$ below represents Americana Airways's routes connecting four cities, J, K, L, and M. Sketch a possible route map.

$$[N] = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 1 & 0 & 2 & 0 \\ 2 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

Mathematics CONNECTION

Graph theory is a branch of mathematics that deals with connections between items. In Exercise 11, a paragraph description of the flight routes could have been made, but a vertex-edge graph of the routes allows you to show the material quickly and clearly. You could also use a graph to diagram a natural gas pipeline, the chemical structure of a molecule, a family tree, or a computer network. The data in a graph can be represented, manipulated mathematically, and further investigated using matrices.



Ozone (O_3)

Review

12. Solve this system using either substitution or elimination.

$$\begin{cases} 5x - 4y = 25 \\ x + y = 3 \end{cases}$$

13. Each slice of pepperoni pizza has approximately 7.4 slices of pepperoni on it, and each slice of supreme pizza has approximately 4.7 slices of pepperoni on it. Write an equation that shows that *p* slices of pepperoni pizza and *s* slices of supreme pizza would have a total of 100 slices of pepperoni.

14. Solve the equation $2x + 3y = 12$ for y and then graph it.

15. **APPLICATION** The table at right shows the number of cellular telephone subscribers in the United States from 1985 to 2000.

a. Create a scatter plot of the data.
b. Find an exponential function to model the data.
c. Use your model to predict the number of subscribers in 2003. Do you think this is a realistic prediction? Why or why not? Do you think an exponential model is appropriate? Why or why not?

Cellular Phone Subscribers

Year	Number of Subscribers	Year	Number of Subscribers
1985	340,000	1993	16,009,000
1986	682,000	1994	24,134,000
1987	1,231,000	1995	33,786,000
1988	2,069,000	1996	44,043,000
1989	3,509,000	1997	55,312,000
1990	5,283,000	1998	69,209,000
1991	7,557,000	1999	86,047,000
1992	11,033,000	2000	109,478,000

(The World Almanac and Book of Facts 2002)

16. **APPLICATION** The equation $y = 20\log\left(\frac{x}{0.00002}\right)$ measures the intensity of a sound as a function of the pressure it creates on the eardrum. The intensity, *y*, is measured in decibels (dB), and the pressure, *x*, is measured in Pascals (Pa).

a. What is the intensity of the sound of a humming refrigerator, if it causes 0.00356 Pa of pressure on the eardrum?
b. A noise that causes 20 Pa of pressure on the eardrum brings severe pain to most people. What is the intensity of this noise?
c. Write the inverse function that measures pressure on the eardrum as a function of intensity of a sound.
d. How much pressure on the eardrum is caused by a 90 dB sound?

LESSON 6.2

Matrix Operations

It is not once nor twice but times without number that the same ideas make their appearance in the world.

ARISTOTLE

A matrix is a compact way of organizing data, similar to a table. Representing data in a matrix instead of a table allows you to perform operations such as addition and multiplication with your data. In this lesson you will see how this is useful.

Consider this problem from Lesson 6.1. Matrix [*A*] represents math, science, and history textbooks sold this week at the main and branch campus bookstores. Matrix [*B*] contains the same information for last week. What are the total sales, by category and location, for both weeks?

$$[A] = \begin{bmatrix} 83 & 33 \\ 65 & 20 \\ 98 & 50 \end{bmatrix} \qquad [B] = \begin{bmatrix} 80 & 25 \\ 65 & 15 \\ 105 & 55 \end{bmatrix}$$

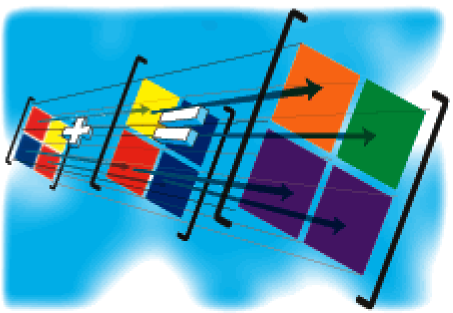
To solve this problem, you add matrices [*A*] and [*B*].

$$83 + 80 = 163$$

$$\begin{bmatrix} 83 & 33 \\ 65 & 20 \\ 98 & 50 \end{bmatrix} + \begin{bmatrix} 80 & 25 \\ 65 & 15 \\ 105 & 55 \end{bmatrix} = \begin{bmatrix} 163 & 58 \\ 130 & 35 \\ 203 & 105 \end{bmatrix}$$

If 83 math books were sold at the main bookstore this week, and 80 math books were sold at the main bookstore last week, a total of 163 math books were sold at the main bookstore for both weeks.

To add two matrices, you simply add corresponding entries. So in order to add (or subtract) two matrices they both must have the same dimensions. The corresponding rows and columns should also have similar interpretations if the results are to make sense. [See **Calculator Note 6A** to learn how to enter matrices into your calculator. **Calculator Note 6B** shows how to perform operations with matrices.]



When you add matrices, you add corresponding entries. This illustration uses the addition of color to show how the addition carries through to the matrix representing the sum.

In Lesson 6.1, you used a matrix to organize the coordinates of the vertices of a triangle. You can use matrix operations to transform a figure such as a triangle just as you transformed the graph of a function.

EXAMPLE A

This matrix represents a triangle.

$$\begin{bmatrix} -3 & 1 & 2 \\ 2 & 3 & -2 \end{bmatrix}$$

a. Graph the triangle and its image after a translation left 3 units. Write a matrix equation to represent the transformation.

b. Describe the transformation represented by this matrix expression:

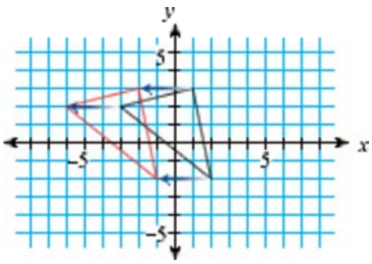
$$\begin{bmatrix} -3 & 1 & 2 \\ 2 & 3 & -2 \end{bmatrix} + \begin{bmatrix} -4 & -4 & -4 \\ -3 & -3 & -3 \end{bmatrix}$$

c. Describe the transformation represented by this matrix expression:

$$2 \cdot \begin{bmatrix} -3 & 1 & 2 \\ 2 & 3 & -2 \end{bmatrix}$$

▸ Solution

The original matrix represents a triangle with vertices (–3, 2), (1, 3), and (2, –2).

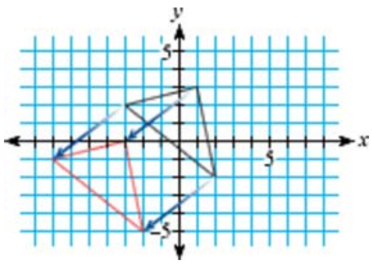


a. After a translation left 3 units, the x-coordinates of the image are reduced by 3. There is no change to the y-coordinates. You can represent this transformation as a subtraction of two matrices.

$$\begin{bmatrix} -3 & 1 & 2 \\ 2 & 3 & -2 \end{bmatrix} - \begin{bmatrix} 3 & 3 & 3 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -6 & -2 & -1 \\ 2 & 3 & -2 \end{bmatrix}$$

b. $$\begin{bmatrix} -3 & 1 & 2 \\ 2 & 3 & -2 \end{bmatrix} + \begin{bmatrix} -4 & -4 & -4 \\ -3 & -3 & -3 \end{bmatrix} = \begin{bmatrix} -7 & -3 & -2 \\ -1 & 0 & -5 \end{bmatrix}$$

This matrix addition represents a translation left 4 units and down 3 units.

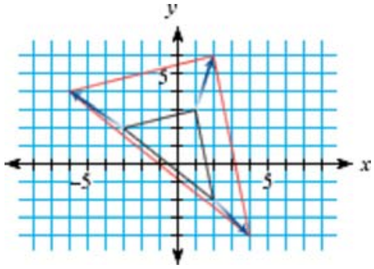


c. $$2 \cdot \begin{bmatrix} -3 & 1 & 2 \\ 2 & 3 & -2 \end{bmatrix} = \begin{bmatrix} -3 & 1 & 2 \\ 2 & 3 & -2 \end{bmatrix} + \begin{bmatrix} -3 & 1 & 2 \\ 2 & 3 & -2 \end{bmatrix} = \begin{bmatrix} -6 & 2 & 4 \\ 4 & 6 & -4 \end{bmatrix}$$

$$2 \cdot \begin{bmatrix} -3 & 1 & 2 \\ 2 & 3 & -2 \end{bmatrix} = \begin{bmatrix} 2 \cdot (-3) & 2 \cdot 1 & 2 \cdot 2 \\ 2 \cdot 2 & 2 \cdot 3 & 2 \cdot (-2) \end{bmatrix} = \begin{bmatrix} -6 & 2 & 4 \\ 4 & 6 & -4 \end{bmatrix}$$

Multiplying a matrix by a number is called **scalar multiplication.** Each entry in the matrix is simply multiplied by the **scalar,** which is 2 in this case.

The resulting matrix represents stretches, both horizontally and vertically, by the scale factor 2. A transformation that stretches or shrinks both horizontally and vertically by the same scale factor is called a **dilation.**

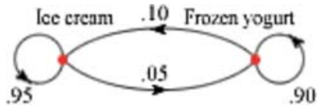


[See Calculator Note 6C to learn how to to use your calculator to graph polygons with matrices.]

Addition and scalar multiplication operate on one entry at a time. The multiplication of two matrices is more involved and uses several entries to find one entry of the answer matrix. Recall this problem from the investigation in Lesson 6.1.

EXAMPLE B

The school cafeteria offers a choice of ice cream or frozen yogurt for dessert once a week. During the first week of school, 220 students choose ice cream and 20 choose frozen yogurt. During each of the following weeks, 10% of the frozen-yogurt eaters switch to ice cream and 5% of the ice-cream eaters switch to frozen yogurt. How many students will choose each dessert in the second week? In the third week?



Solution

You can use this matrix equation to find the answer for the second week:

$$\begin{bmatrix} 220 & 20 \end{bmatrix} \begin{bmatrix} .95 & .05 \\ .10 & .90 \end{bmatrix} = [\textit{ice cream} \quad \textit{frozen yogurt}]$$

The initial matrix, $[A] = [220 \quad 20]$, represents the original numbers of ice-cream eaters and frozen-yogurt eaters.

In the transition matrix $[B] = \begin{bmatrix} .95 & .05 \\ .10 & .90 \end{bmatrix}$, the top row represents the transitions in the current number of ice-cream eaters, and the bottom row represents the transitions in the current number of frozen-yogurt eaters.

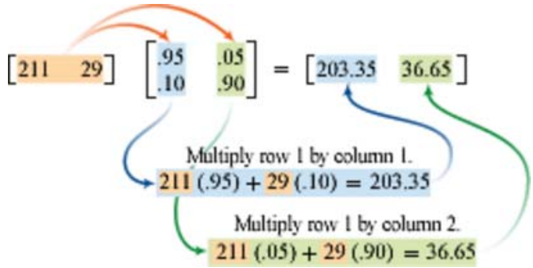
You can define matrix multiplication by looking at how you calculate the numbers for the second week. The second week's number of ice-cream eaters will be 220(.95) + 20(.10), or 211 students, because 95% of the 220 original ice-cream eaters don't switch and 10% of the 20 original frozen-yogurt eaters switch to ice cream. In effect, you multiply the two entries in row 1 of [A] by the two entries in column 1 of [B] and add the products. The result, 211, is entry c_{11} in the answer matrix, [C].

Initial matrix		Transition matrix		Answer matrix
$[A]$	$\cdot$	$[B]$	$=$	$[C]$
$[220 \quad 20]$		$\begin{bmatrix} .95 & .05 \\ .10 & .90 \end{bmatrix}$	$=$	$[211 \quad \textit{frozen yogurt}]$

Likewise, the second week's number of frozen-yogurt eaters will be 220(.05) + 20(.90), or 29 students, because 5% of the ice-cream eaters switch to frozen yogurt and 90% of the frozen-yogurt eaters don't switch. The number of frozen-yogurt eaters in the second week is the sum of the products of the entries in row 1 of [*A*] and column 2 of [*B*]. The answer, 29, is entry c_{12} in the answer matrix, [*C*].

$$\begin{bmatrix} 220 & 20 \end{bmatrix} \begin{bmatrix} .95 & .05 \\ .10 & .90 \end{bmatrix} = \begin{bmatrix} 211 & 29 \end{bmatrix}$$

To get the numbers for the third week, multiply the result of your previous calculations by the transition matrix again.



Approximately 203 students will choose ice cream and 37 will choose frozen yogurt in the next week.

You can continue multiplying to find the numbers in the fourth week, the fifth week, and so on. [▶ Revisit **Calculator Note 6B** to learn how to multiply matrices on your calculator. ◀]

In the investigation you will model a real-world situation with matrices. You'll also practice multiplying matrices.

Investigation
Find Your Place

In this investigation you will simulate the weekly movement of rental cars between cities and analyze the results.

Each person represents a rental car starting at City A, City B, or City C.

Step 1 In a table, record the number of cars that start in each city. Follow the Procedure Note to simulate the movement of cars.

> **Procedure Note**
>
> **Rental Car Simulation**
>
> **1.** Use your calculator to generate a random number, x, between 0 and 1. [▶ See **Calculator Note 1L** to learn how to generate random numbers. ◀] Determine your location for next week as follows:
>
> **a.** If you are at City A, move to City B if $x \le .2$, move to City C if $.2 < x \le .7$, or stay at City A if $x > .7$.
>
> **b.** If you are at City B, move to City A if $x \le .5$, or stay at City B if $x > .5$.
>
> **c.** If you are at City C, move to City B if $x \le .1$, move to City A if $.1 < x \le .3$, or stay at City C if $x > .3$.
>
> **2.** Record the number of cars in each city in your table. Repeat the simulation up to 10 times. Each time, record the number of cars in each city.

Step 2 Work with your group to make a transition diagram and a transition matrix that represent the rules of the simulation.

Step 3 Write an initial condition matrix for the starting quantities at each city. Then, show how to multiply the initial condition matrix and the transition matrix for the first transition. How do these theoretical results for week 1 compare with the experimental data from your simulation?

Step 4 Use your calculator to find the theoretical number of cars in each city for the next four weeks. Find the theoretical long-run values of the number of cars in each city.

Step 5 Compare these results with the experimental values in your table. If they are not similar, explain why.

Just as only some matrices can be added (those with the same dimensions), only some matrices can be multiplied. Example C and Exercise 3 will help you explore the kinds of matrices that can be multiplied.

American artist Robert Silvers (b 1968) combined thousands of worldwide money images in a matrix-like arrangement to create this piece titled *Washington.*

EXAMPLE C

Consider this product.

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -3 & 1 & 2 \\ 2 & 3 & -2 \end{bmatrix}$$

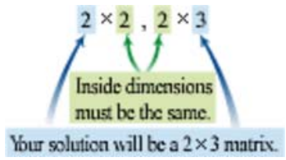
a. Determine the dimensions of the answer to this product.

b. Describe how to calculate entries in the answer.

▸ Solution

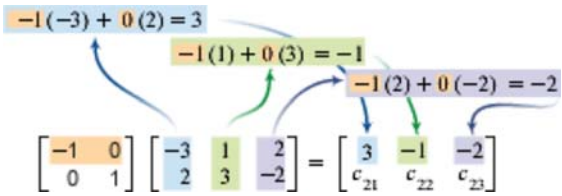
a. To multiply two matrices, you multiply each entry in a row of the first matrix by each entry in a column of the second matrix.

You can multiply a 2 × 2 matrix by a 2 × 3 matrix because the inside dimensions are
the same- the 2 row entries match up with the
2 column entries.
The outside dimensions tell you the dimensions of your answer.
The answer to this product has dimensions 2 × 3.

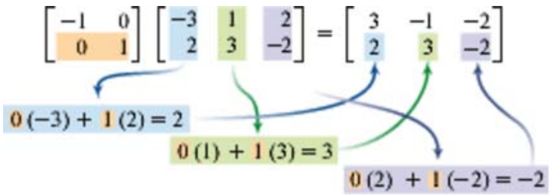


$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -3 & 1 & 2 \\ 2 & 3 & -2 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{bmatrix}$$

b. To find the values of entries in the first row of your solution matrix, you add the products of the entries in the first row of the first matrix and the entries in the columns of the second matrix.



To find the values of entries in the second row of your solution matrix, you add the products of the entries in the second row of the first matrix and the entries in the columns of the second matrix.



The product is

$$\begin{bmatrix} 3 & -1 & -2 \\ 2 & 3 & -2 \end{bmatrix}$$

The following definitions review the matrix operations you've learned in this lesson.

Matrix Operations

Matrix Addition

To add matrices, you add corresponding entries.

$$\begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} -1 & 2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ 0 & 4 \end{bmatrix}$$

$$-1 + 1 = 0$$

You can add only matrices that have the same dimension.

Scalar Multiplication

To multiply a scalar by a matrix, you multiply the scalar by each value in a matrix.

$$3 \cdot \begin{bmatrix} 2 & -1 \\ 1 & 3 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 6 & -3 \\ 3 & 9 \\ 0 & -6 \end{bmatrix}$$

$$3(-2) = -6$$

Matrix Multiplication

To multiply two matrices, [*A*] and [*B*], you multiply each entry in a row of matrix [*A*] by corresponding entries in a column of matrix [*B*].

$$\begin{bmatrix} 0 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & -2 & 4 \\ 0 & -3 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 3 & -5 \\ 6 & -7 & 13 \end{bmatrix}$$

$$2(-2) + 1(-3) = -7$$

Entry c_{ij} in the answer matrix, [*C*], represents the sum of the products of each entry in row *i* of the first matrix and the entry in the corresponding position in column *j* of the second matrix. The number of entries in a row of matrix [*A*] must equal the number of entries in a column of matrix [*B*]. That is, the inside dimensions must be equal. The answer matrix will have the same number of rows as matrix [*A*] and the same number of columns as matrix [*B*], or the outside dimensions.

EXERCISES

Practice Your Skills

1. Look back at the calculations in Example B. Calculate how many students will choose each dessert in the fourth week by multiplying these matrices:

$$\begin{bmatrix} 203.35 & 36.65 \end{bmatrix} \begin{bmatrix} .95 & .05 \\ .10 & .90 \end{bmatrix} = \begin{bmatrix} \textit{ice cream} & \textit{frozen yogurt} \end{bmatrix}$$

2. Find the missing values.

a. $[13 \quad 23] + [-6 \quad 31] = [x \quad y]$

b. $\begin{bmatrix} .90 & .10 \\ .05 & .95 \end{bmatrix}\begin{bmatrix} .90 & .10 \\ .05 & .95 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$

c. $\begin{bmatrix} 18 & -23 \\ 5.4 & 32.2 \end{bmatrix} + \begin{bmatrix} -2.4 & 12.2 \\ 5.3 & 10 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

d. $10 \cdot \begin{bmatrix} 18 & -23 \\ 5.4 & 32.2 \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$

e. $\begin{bmatrix} 7 & -4 \\ 18 & 28 \end{bmatrix} + 5 \cdot \begin{bmatrix} -2.4 & 12.2 \\ 5.3 & 10 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

American painter Chuck Close (b 1940) creates photo realistic portraits by painting a matrix-like grid of rectangular cells. Close is a quadriplegic and paints with a mouth brush. This portrait is from 1992.
Janet by Chuck Close, oil on canvas, 102 × 84 in.

3. Perform matrix arithmetic in 3a-f. If a particular operation is impossible, explain why.

a. $\begin{bmatrix} 1 & 2 \\ 3 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -3 & -1 & 2 \\ 5 & 2 & -1 \end{bmatrix}$

b. $\begin{bmatrix} 1 & -2 \\ 6 & 3 \end{bmatrix} + \begin{bmatrix} -3 & 7 \\ 2 & 4 \end{bmatrix}$

c. $\begin{bmatrix} 5 & -2 & 7 \end{bmatrix}\begin{bmatrix} -2 & 3 \\ -1 & 0 \\ 3 & 2 \end{bmatrix}$

d. $\begin{bmatrix} 3 & -8 & 10 & 2 \\ -1 & 2 & 3 & 4 \end{bmatrix}\begin{bmatrix} 2 & -5 & 3 & 12 \\ 8 & -4 & 0 & 2 \end{bmatrix}$

e. $\begin{bmatrix} 3 & 6 \\ -4 & 1 \end{bmatrix} - \begin{bmatrix} -1 & 7 \\ -8 & 3 \end{bmatrix}$

f. $\begin{bmatrix} 4 & 11 \\ 7 & 3 \\ 4 & 2 \end{bmatrix} + \begin{bmatrix} 3 & -2 & 7 \\ 5 & 0 & 2 \end{bmatrix}$

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4. Find matrix $[B]$ such that

$$\begin{bmatrix} 8 & -5 & 4.5 \\ -6 & 9.5 & 5 \end{bmatrix} - [B] = \begin{bmatrix} 5 & -1 & 2 \\ -4 & 3.5 & 1 \end{bmatrix}$$

Reason and Apply

5. This matrix represents a triangle:

$$\begin{bmatrix} -3 & 1 & 2 \\ 2 & 3 & -2 \end{bmatrix}$$

a. Graph the triangle.

b. Find the result of this matrix multiplication:

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & -1 & -2 \\ 2 & 3 & -2 \end{bmatrix}$$

c. Graph the image represented by the matrix in 5b.

d. Describe the transformation.

6. Find matrix [*A*] and matrix [*C*] such that the triangle represented by

$$[T] = \begin{bmatrix} -3 & 1 & 2 \\ 2 & 3 & -2 \end{bmatrix}$$

matrix [*T*] is reflected across the *x*-axis.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} -3 & 1 & 2 \\ 2 & 3 & -2 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{bmatrix}$$

7. Of two-car families in a small city, 88% remain two-car families in the following year and 12% become one-car families in the following year. Of one-car families, 72% remain one-car families and 28% become two-car families. Suppose these trends continue for a few years. At present, 4800 families have one car and 4200 have two cars.

a. Draw a transition diagram that displays this information.

b. What matrix represents the present situation? Let a_{11} represent one-car families that remain one-car families.

c. Write a transition matrix that represents the same information as your transition diagram.

d. Write a matrix equation to find the numbers of one-car and two-car families one year from now.

e. Find the numbers of one-car and two-car families two years from now.

8. ***Mini-Investigation*** Enter these matrices into your calculator.

$$[A] = \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix} \quad [B] = \begin{bmatrix} 3 & 4 \\ 0 & -2 \end{bmatrix} \quad [C] = \begin{bmatrix} -2 & 3 & 0 \\ -1 & 5 & 4 \end{bmatrix} \quad [D] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

a. Find [*A*][*B*] and [*B*][*A*]. Are they the same?

b. Find [*A*][*C*] and [*C*][*A*]. Are they the same? What do you notice?

c. Find [*A*][*D*] and [*D*][*A*]. Are they the same? What do you notice?

d. Is matrix multiplication commutative? That is, does order matter?

9. Find the missing values.

a. $\begin{bmatrix} 2 & a \\ b & -1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 19 \\ 17 \end{bmatrix}$

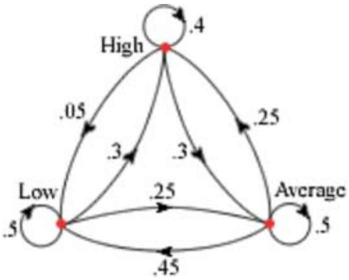
b. $\begin{bmatrix} a & -2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ b \end{bmatrix} = \begin{bmatrix} -29 \\ -5 \end{bmatrix}$

10. Recall the ice cream and frozen yogurt problem from Example B. Enter these matrices into your calculator, and use them to find the long-run values for the number of students who choose ice cream and the number of those who choose frozen yogurt. Explain why your answer makes sense.

$[A] = [220 \quad 20]$ $\qquad [B] = \begin{bmatrix} .95 & .05 \\ .10 & .90 \end{bmatrix}$

11. A spider is in a building with three rooms. The spider moves from room to room by choosing a door at random. If the spider starts in room 1 initially, what is the probability that it will be in room 1 again after four room changes? What happens to the probabilities in the long run?

12. **APPLICATION** A researcher studies the birth weights of women and their daughters. The weights were split into three categories: low (below 6 lb), average (between 6 and 8 lb), and high (above 8 lb). This transition diagram shows how birth weights changed from mother to daughter.

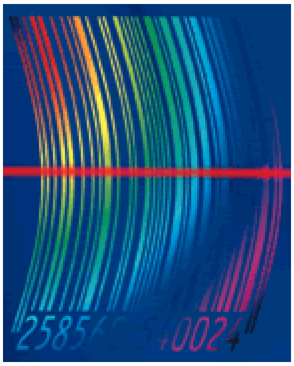


a. Write a transition matrix that represents the same information as the diagram. Put the rows and columns in the order low, average, high.

b. Assume the changes in birth weights can be applied to any generation. If, in the initial generation of women, 25% had birth weights in the low category, 60% in the average category, and 15% in the high category, what were the percentages after one generation? After two generations? After three generations? In the long run?

Consumer CONNECTION

You can find a Universal Product Code, or UPC, symbol on almost every mass-produced product. The symbol consists of vertical lines with a sequence of 12 numbers below them. The lines are readable to a scanning device as numbers. The first six digits represent the manufacturer and the next five represent the specific product. The last digit is a check digit so that when the item is scanned, the computer can verify the correctness of the number before searching its database to get the price. In order to verify the code, each digit in an odd position is multiplied by 3. These products are then added with the digits in the even positions (including the check digit). The check digit is chosen so that this sum is divisible by 10.



13. **APPLICATION** Read the Consumer Connection about Universal Product Codes and answer these questions.

a. Write a 12×1 matrix that can be multiplied on the left by a UPC to find the sum of each digit in an even position and 3 times each digit in an odd position.

b. Use the matrix from 13a to check the following four UPCs. Which one(s) are valid?

$$\begin{bmatrix} 0 & 3 & 6 & 2 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 5 \\ 0 & 7 & 6 & 1 & 0 & 7 & 0 & 2 & 2 & 3 & 3 & 6 \\ 0 & 7 & 4 & 2 & 2 & 0 & 0 & 0 & 2 & 9 & 1 & 8 \\ 0 & 8 & 5 & 3 & 9 & 1 & 7 & 8 & 6 & 2 & 2 & 1 \end{bmatrix}$$

c. For the invalid UPC(s), what should the check digit have been so that it is a valid code?

Review

14. ***Mini-Investigation*** A system of equations that has at least one solution is called **consistent**. A system of equations that has no solutions is called **inconsistent.** A system with infinitely many solutions is called **dependent.** A system of equations that has exactly one solution is called **independent.** Follow the steps in 14a-g to make some discoveries about inconsistent and dependent systems.

a. Graph each of the following systems of linear equations. Use your graphs to identify each system as consistent, inconsistent, dependent, and/or independent.

i. $\begin{cases} y = 0.7x + 8 \\ y = 1.1x - 7 \end{cases}$ **ii.** $\begin{cases} y = \frac{3}{4}x - 4 \\ y = 0.75x + 3 \end{cases}$ **iii.** $\begin{cases} 4x + 6y = 9 \\ 1.2x + 1.8y = 4.7 \end{cases}$

iv. $\begin{cases} \frac{3}{4}x - \frac{1}{2}y = 4 \\ 0.75x + 0.5y = 3 \end{cases}$ **v.** $\begin{cases} y = 1.2x + 3 \\ y = 1.2x - 1 \end{cases}$ **vi.** $\begin{cases} y = \frac{1}{4}(2x - 1) \\ y = 0.5x - 0.25 \end{cases}$

vii. $\begin{cases} 4x + 6y = 9 \\ 1.2x + 1.8y = 2.7 \end{cases}$ **viii.** $\begin{cases} \frac{3}{5}x - \frac{2}{5}y = 3 \\ 0.6x + 0.4y = 3 \end{cases}$ **ix.** $\begin{cases} x + y = 1 \\ 2x + 2y = 2 \end{cases}$

b. Describe the graphs of the equations of the inconsistent systems.

c. Try to solve each inconsistent system by substitution or by elimination. Show your steps. Describe the outcome of your attempts.

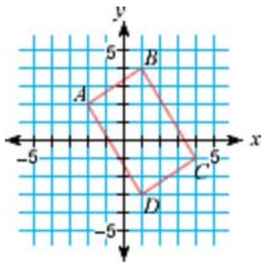
d. How can you recognize an inconsistent linear system without graphing it?

e. Describe the graphs of equations of consistent and dependent systems.

f. Try to solve each consistent and dependent system by substitution or by elimination. Show your steps. Describe the outcome of your attempt.

g. How can you recognize a consistent and dependent linear system without graphing it?

15. For each segment shown in the figure at right, write an equation in point-slope form for the line that contains the segment. Check your equations by graphing them on your calculator.



16. If $\log_p x = a$ and $\log_p y = b$, find

a. $\log_p xy$ **b.** $\log_p x^3$ **c.** $\log_p \frac{y^2}{x}$

d. $\log_{p^2} y$ **e.** $\log_p \sqrt{x}$ **f.** $\log_m xy$

17. Solve this system of equations for x, y, and z.

$$\begin{cases} x + 2y + z = 0 \\ 3x - 4y + 5z = -11 \\ -2x - 8y - 3z = 1 \end{cases}$$

LESSON

6.3

Row Reduction Method

Rather than denying problems, focus inventively, intentionally on what solutions might look or feel like . . .

MARSHA SINETAR

In Chapter 3, you learned how to solve systems of equations using elimination. You added equations, sometimes first multiplying both sides by a convenient factor, to reduce the system to an equation in one variable. In this lesson you will learn how to use matrices to simplify this elimination method for solving systems of equations, especially when you have more than two variables.

Any system of equations in standard form can be written as a matrix equation. For example

$$\begin{cases} 2x + y = 5 \\ 5x + 3y = 13 \end{cases}$$ The original system.

$$\begin{bmatrix} 2x + y \\ 5x + 3y \end{bmatrix} = \begin{bmatrix} 5 \\ 13 \end{bmatrix}$$ Rewrite with matrices.

$$\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 13 \end{bmatrix}$$ The product $\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ is equivalent to $\begin{bmatrix} 2x + y \\ 5x + 3y \end{bmatrix}$.

You can also write the system as an **augmented matrix,** which is a single matrix that contains a column for the coefficients of each variable and a final column for the constant terms.

$$\begin{cases} 2x + y = 5 \\ 5x + 3y = 13 \end{cases} \rightarrow \left[\begin{array}{cc|c} 2 & 1 & 5 \\ 5 & 3 & 13 \end{array}\right]$$

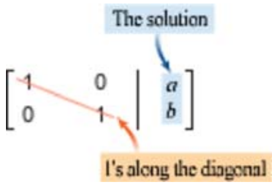
In this piece by Belgian painter René Magritte (1898-1967), a man appears in one frame, but he is "eliminated" from the others. *Man with a Newspaper (1928) by René Magritte, oil on canvas*

You can use the augmented matrix to carry out a process similar to elimination.

The **row reduction method** transforms an augmented matrix into a solution matrix. Instead of combining equations and multiples of equations until you are left with an equation in one variable, you add multiples of rows to other rows until you obtain the solution matrix. A solution matrix contains the solution to the system in the last column. The rest of the matrix consists of 1's along the main diagonal and 0's above and below it.

This augmented matrix represents the system

$$\begin{cases} 1x + 0y = a \\ 0x + 1y = b \end{cases}$$, or $x = a$ and $y = b$.



This matrix is in **reduced row-echelon form** because each row is reduced to a 1 and a solution, and the rest of the matrix entries are 0's. The 1's are in echelon, or step, form. The ordered pair (a, b) is the solution to the system.

An augmented matrix represents a system of equations, so the same rules apply to row operations in a matrix as to equations in a system of equations.

Row Operations in a Matrix

- You can multiply (or divide) all numbers in a row by a nonzero number.
- You can add all numbers in a row to corresponding numbers in another row.
- You can add a multiple of the numbers in one row to the corresponding numbers in another row.
- You can exchange two rows.

EXAMPLE A

Solve this system of equations.

$$\begin{cases} 2x + y = 5 \\ 5x + 3y = 13 \end{cases}$$

Solution

You can solve the system using matrices or equations. Let's compare the row reduction method using matrices with the elimination method using equations.

Because the equations are in standard form, you can copy the coefficients and constants from each equation into corresponding rows of the augmented matrix.

$$\begin{cases} 2x + y = 5 \\ 5x + 3y = 13 \end{cases} \rightarrow \left[\begin{array}{cc|c} 2 & 1 & 5 \\ 5 & 3 & 13 \end{array}\right]$$

Let's call this augmented matrix $[M]$. Using only the elementary row operations, you can transform this matrix into the solution matrix. You need both m_{21} and m_{12} to be 0, and you need both m_{11} and m_{22} to be 1.

Add –2.5 times row 1 to row 2 to get 0 for m_{21}.

$$\left[\begin{array}{cc|c} 2 & 1 & 5 \\ 0 & 0.5 & 0.5 \end{array}\right]$$

Multiply equation 1 by –2.5 and add to row 2 to eliminate x.

$$\begin{array}{rcr} -5x - 2.5y & = & -12.5 \\ 5x + 3y & = & 13 \\ \hline 0.5y & = & 0.5 \end{array}$$

Multiply row 2 by 2 to change m_{22} to 1.

$$\left[\begin{array}{cc|c} 2 & 1 & 5 \\ 0 & 1 & 1 \end{array}\right]$$

Multiply the equation by 2 to find y.

$$y = 1$$

Add –1 times row 2 to row 1 to get 0 for m_{12}.

$$\left[\begin{array}{cc|c} 2 & 0 & 4 \\ 0 & 1 & 1 \end{array}\right]$$

Multiply –1 by this new equation, and add the result to the first equation to eliminate y.

$$\begin{array}{rcr} 2x + y & = & 5 \\ -y & = & -1 \\ \hline 2x & = & 4 \end{array}$$

Multiply row 1 by 0.5.

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right]$$

Multiply the equation by 0.5 to find x.

$$x = 2$$

The last column of the solution matrix indicates that the solution to the system is (2, 1).

You can represent row operations symbolically. For example, you can use R_1 and R_2 to represent the two rows of a matrix, as in Example A, and show the steps this way:

Original matrix $\left[\begin{array}{cc|c} 2 & 1 & 5 \\ 5 & 3 & 13 \end{array}\right]$

$-2.5R_1 + R_2 \rightarrow R_2$ Add −2.5 times row 1 to row 2.

$\left[\begin{array}{cc|c} 2 & 1 & 5 \\ 0 & 0.5 & 0.5 \end{array}\right]$

$2R_2 \rightarrow R_2$ Multiply row 2 by 2.

$\left[\begin{array}{cc|c} 2 & 1 & 5 \\ 0 & 1 & 1 \end{array}\right]$

$-R_2 + R_1 \rightarrow R_1$ Add −1 times row 2 to row 1.

$\left[\begin{array}{cc|c} 2 & 0 & 4 \\ 0 & 1 & 1 \end{array}\right]$

$0.5R_1 \rightarrow R_1$ Multiply row 1 by 0.5.

Solution matrix $\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right]$

Investigation
League Play

The number of games a soccer league must schedule depends on the number of teams playing in that league. This table shows the number of games required for each team in a league to play every other team twice, once at each team's home field. In this investigation you will find a function that describes the number of games for any number of teams.

Number of teams	1	2	3	4	5	6	7
Number of games	0	2	6	12	20	30	42

Step 1 Make a scatter plot of these data. Let x represent the number of teams, and let y represent the number of games. Describe the graph. Is it linear?

Step 2 Based on the shape of the graph, the equation could be quadratic. You can write a quadratic equation in the form $y = ax^2 + bx + c$. You can use each pair of values in the table to write an equation by substituting x and y. Create three equations with the variables a, b, and c by substituting any three pairs of coordinates from the above table. For example, the point (5, 20) creates the equation $a(5)^2 + b(5) + c = 20$, or $25a + 5b + c = 20$.

Step 3 You can now solve for the coefficients a, b, and c. Write a 3×4 augmented matrix for your system of three equations.

Step 4 Find the row operations that will give 0 for m_{13} and m_{23}. Describe the operations and write them symbolically using R_1, R_2, and R_3.

Step 5 Find row operations that give 0's in the other nondiagonal entries of your augmented matrix. Write all the row operations in terms of R_1, R_2, and R_3.

Step 6 Find row operations that give 1's along the main diagonal. Your matrix should now be in the form

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & k_1 \\ 0 & 1 & 0 & k_2 \\ 0 & 0 & 1 & k_3 \end{array}\right]$$

What does this mean about your equation, $y = ax^2 + bx + c$? Verify that you have the correct values for a, b, and c.

Step 7 Write a summary of the steps you followed to solve this problem. Describe any problems you ran into and any tricks or shortcuts you found.

Equations such as $2x + y = 5$ or $y = 3x + 4$ are called linear equations because their graphs in the coordinate plane are always lines. You may also notice that in linear equations the highest power of x or y is 1 and that x and y are never multiplied together. Equations in three variables such as $2x + y + 3z = 12$, where the highest power is 1, are also called linear equations. In the investigation you used an augmented matrix to solve a system of three linear equations in three variables. Here's another example of solving a larger system with the help of matrices.

EXAMPLE B

The junior class treasurer is totaling the sales and receipts from the last book sale. She has 50 receipts for sales of three different titles of books priced at $14.00, $18.50, and $23.25. She has a total of $909.00 and knows that 22 more of the $18.50 books sold than the $23.25 books. How many of each book were sold?

▶ Solution

The numbers sold of the three different book titles are unknown, so you can assign three variables.

x = the number of $14.00 books
y = the number of $18.50 books
z = the number of $23.25 books

Based on the information in the problem, write a system of three linear equations. The system can also be written as an augmented matrix.

$$\begin{cases} x + y + z = 50 \\ 14x + 18.50y + 23.25z = 909 \\ y - z = 22 \end{cases} \quad \text{or} \quad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 50 \\ 14 & 18.5 & 23.25 & 909 \\ 0 & 1 & -1 & 22 \end{array}\right]$$

Here is one possible sequence of row operations to obtain a solution matrix. Try these row operations to see how they transform the augmented matrix into reduced row-echelon form. [▶ See **Calculator Note 6D** to learn how to do row operations on your calculator. ◀]

$$\left.\begin{array}{l} -14R_1 + R_2 \rightarrow R_2 \\ -R_3 + R_1 \rightarrow R_1 \\ -4.5R_3 + R_2 \rightarrow R_2 \\ \dfrac{R_2}{13.75} \rightarrow R_2 \\ R_2 \leftrightarrow R_3 \\ R_3 + R_2 \rightarrow R_2 \\ -2R_3 + R_1 \rightarrow R_1 \end{array}\right\}$$

This sequence of row operations gives this reduced row-echelon matrix.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 12 \\ 0 & 1 & 0 & 30 \\ 0 & 0 & 1 & 8 \end{array}\right]$$

This means that 12 of the $14.00 books, 30 of the $18.50 books, and 8 of the $23.25 books were sold.

Some systems of equations have no solution, and others have infinitely many solutions. Likewise, not all augmented matrices can be reduced to row-echelon form. An entire row of 0's means that one equation is equivalent to another; therefore, not enough information was given to find a single solution, so there are infinitely many solutions. If, on the other hand, an entire row reduces to 0's, except for a nonzero constant in the last entry, no solution exists because a set of 0 coefficients cannot result in a nonzero constant on the right side of the equation.

EXERCISES

Practice Your Skills

1. Write a system of equations for each augmented matrix.

a. $\left[\begin{array}{cc|c} 2 & 5 & 8 \\ 4 & -1 & 6 \end{array}\right]$

b. $\left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 1 & 2 & -3 & 1 \\ 2 & 1 & -1 & 2 \end{array}\right]$

2. Write an augmented matrix for each system.

a. $\begin{cases} x + 2y - z = 1 \\ 2x - y + 3z = 2 \\ 2x + y + z = -1 \end{cases}$

b. $\begin{cases} 2x + y - z = 12 \\ 2x + z = 4 \\ 2x - y + 3z = -4 \end{cases}$

3. Perform each row operation on this matrix.

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 1 & 2 & -3 & 1 \\ 2 & 1 & -1 & 2 \end{array}\right]$$

a. $-R_1 + R_2 \longrightarrow R_2$

b. $-2R_1 + R_3 \longrightarrow R_3$

4. Give the missing row operation or matrix in the table.

	Description	Matrix
a.	The original system. $\begin{cases} 2x + 5y = 8 \\ 4x - y = 6 \end{cases}$	$\left[\begin{array}{cc\|c} & & \\ & & \end{array}\right]$
b.	$-2R_1 + R_2 \longrightarrow R_2$	$\left[\begin{array}{cc\|c} & & \\ & & \end{array}\right]$
c.		$\left[\begin{array}{cc\|c} 2 & 5 & 8 \\ 0 & 1 & \frac{10}{11} \end{array}\right]$
d.		$\left[\begin{array}{cc\|c} 2 & 0 & \frac{38}{11} \\ 0 & 1 & \frac{10}{11} \end{array}\right]$
e.		$\left[\begin{array}{cc\|c} 1 & 0 & \frac{19}{11} \\ 0 & 1 & \frac{10}{11} \end{array}\right]$

Reason and Apply

5. Rewrite each system of equations as an augmented matrix. If possible, transform the matrix into its reduced row-echelon form using row operations on your calculator.

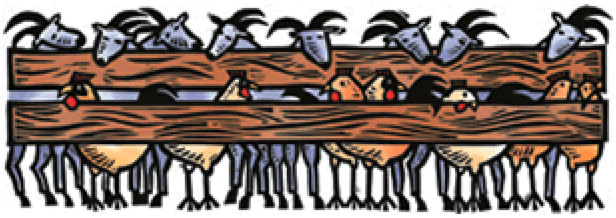
a. $\begin{cases} x + 2y + 3z = 5 \\ 2x + 3y + 2z = 2 \\ -x - 2y - 4z = -1 \end{cases}$

b. $\begin{cases} -x + 3y - z = 4 \\ 2z = x + y \\ 2.2y + 2.2z = 2.2 \end{cases}$

c. $\begin{cases} 3x - y + z = 7 \\ x - 2y + 5z = 1 \\ 6x - 2y + 2z = 14 \end{cases}$

d. $\begin{cases} 3x - y + z = 5 \\ x - 2y + 5z = 1 \\ 6x - 2y + 2z = 14 \end{cases}$

6. A farmer raises only goats and chickens on his farm. All together he has 47 animals, and they have a total of 118 legs.



a. Write a system of equations and an augmented matrix. How many of each animal does he have? [► See Calculator Note 6E to learn how to transform a matrix to reduced row-echelon form on your calculator. ◄]

b. The farmer's neighbor also has goats and chickens. She reports having 118 animals with a total of 47 legs. Write a system of equations and an augmented matrix. How many of each animal does she have?

7. The largest angle of a triangle is 4° more than twice the smallest angle. The smallest angle is 24° less than the midsize angle. What are the measures of the three angles?

8. **APPLICATION** The amount of merchandise that is available for sale is called supply. The amount of merchandise that consumers want to buy is called demand. Supply and demand are in equilibrium, or balance, when a price is found that makes supply and demand equal.

Suppose the following data represent supply and demand for a drink manufacturer.

Supply for Mega-Fruit

Price (cents/gal)	Quantity (millions of gal)
80	1304.4
90	2894
100	4483.6
110	6073.2
120	7662.8

Demand for Mega-Fruit

Price (cents/gal)	Quantity (millions of gal)
80	3268.47
90	2724.87
100	2181.27
110	1637.67
120	1094.07

a. Find linear models for the supply and demand.

b. Find the equilibrium point graphically.

c. Write the supply and demand equations from 8a as a system in an augmented matrix. Use row reduction to verify your answer to 8b.

Economics CONNECTION

Supply and demand are affected by many things. The supply may be affected by the price of the merchandise, the cost of making it, or unexpected events that affect supply, like drought or hurricanes. Demand may be affected by the price, the income level of the consumer, or consumer tastes. An increased price may slow the purchase of the product and thus also increase supply. The stock market illustrates how prices are determined through the interaction of supply and demand in an auction-like environment.

In New York City, January 1996, grocery store shelves emptied as residents stocked up for a severe snowstorm. Unexpected events like this can cause high demand and deplete supply.

9. Find a, b, and c such that the graph of $y = ax^2 + bx + c$ passes through the points (1, 3), (4, 24), and (–2, 18).

10. The yearbook staff sells ads in three sizes. The full-page ads sell for \$200, the half-page ads sell for \$125, and the business-card-size ads sell for \$20. All together they earned \$1715 from 22 ads. There were four times as many business-card-size ads sold as full-page ads. How many of each ad type did they sell?

Review

11. **APPLICATION** The Life is a Dance troupe has two choices in how it will be paid for its next series of performances. The first option is to receive \$12,500 for the series plus 5% of all ticket sales. The second option is \$6,800 for the series plus 15% of ticket sales. The company will perform three consecutive nights in a hall that seats 2,200 people. All tickets will cost \$12.

Honduran-American dancer Homer Avila performed a solo work titled *Not/Without Words* in February 2002, one year after losing his leg and part of his hip to cancer.

a. How much will the troupe receive under each plan if a total of 3500 tickets are sold for all three performances?

b. Write an equation that gives the amount the troupe will receive under the first plan for any number of tickets sold.

c. Write an equation that gives the amount the troupe will receive under the second plan for any number of tickets sold.

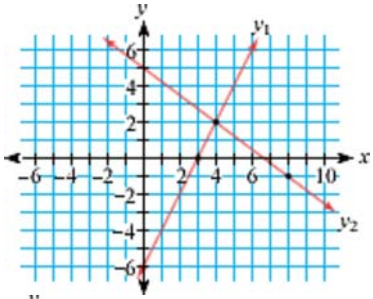
d. How many tickets must the troupe sell for the second plan to be the better choice?

e. Which plan should the troupe choose? Justify your choice.

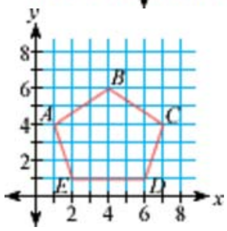
12. Consider this graph of a system of two linear equations.

a. What is the solution to this system?

b. Write equations for the two lines.



13. For each segment shown in the pentagon at right, write an equation in point-slope form for the line that contains the segment. Check your equations by graphing them on your calculator.



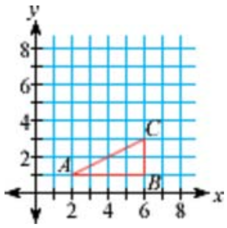
14. Consider the graph of $\triangle ABC$.

a. Represent $\triangle ABC$ with a matrix $[M]$.

b. Find each product and graph the image of the triangle represented by the result.

i. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}[M]$

ii. $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}[M]$



IMPROVING YOUR VISUAL THINKING SKILLS

Intersection of Planes

Graphically, a system of two linear equations in two variables can be represented by two lines. If the lines intersect, the point of intersection is the solution and the system is called consistent and independent. If the lines are parallel, they never intersect, there is no solution, and the system is called inconsistent. If the lines are the same, there are infinitely many solutions, and the system is called consistent and dependent.

An equation such as $3x + 2y + 6z = 12$ is also called a linear equation because the highest power of any variable is 1. But, because there are three variables, the graph of this equation is a plane.

Graphically, a system of three linear equations in three variables can be represented by three planes. Sketch all the possible outcomes for the graphs of three planes. Classify each outcome as consistent, inconsistent, dependent, and/or independent.

LESSON

6.4 Solving Systems with Inverse Matrices

Things that oppose each other also complement each other.

CHINESE SAYING

Consider the equation $ax = b$. To solve for x, you multiply both sides of the equation by $\frac{1}{a}$, the **multiplicative inverse** of a. The multiplicative inverse of a nonzero number, such as 2.25, is the number that you can multiply by 2.25 to get 1. Also, the number 1 is the **multiplicative identity** because any number multiplied by 1 remains unchanged.

Similarly, to solve a system by using matrices, you can use an **inverse matrix.** If an inverse matrix exists, then when you multiply it by the system matrix you will get the matrix equivalent of 1, which is called the **identity matrix.** Any square matrix multiplied on either side by the identity matrix of the same dimensions remains unchanged, just as any number multiplied by 1 remains unchanged. In Example A, you will first use this multiplicative identity to find a 2×2 identity matrix.

EXAMPLE A Find an identity matrix for $\begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$.

Solution

You want to find a matrix, $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, that satisfies the definition of the identity matrix.

$$\begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$$

Multiplying by an identity matrix leaves the matrix unchanged.

$$\begin{bmatrix} 2a+c & 2b+d \\ 4a+3c & 4b+3d \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$$

Multiply the left side.

Because the two matrices are equal, their entries must be equal. Setting corresponding entries equal produces these equations:

$2a + c = 2$ $\quad$ $2b + d = 1$

$4a + 3c = 4$ $\quad$ $4b + 3d = 3$

You can treat these as two systems of equations. Use substitution, elimination, or an augmented matrix to solve each system.

$$\begin{cases} 2a + c = 2 \\ 4a + 3c = 4 \end{cases}$$

A system that can be solved for a and c.

$$\begin{array}{rcr} -6a - 3c & = & -6 \\ 4a + 3c & = & 4 \\ \hline -2a & = & -2 \\ a & = & 1 \end{array}$$

Multiply the first equation by –3.
Add the equations to eliminate c.
Solve for a.

$2(1) + c = 2$

$c = 0$

Substitute 1 for a in the first equation to find c.
Solve for c.

This system gives $a = 1$ and $c = 0$. You can use a similar procedure to find that $b = 0$ and $d = 1$.

The 2×2 identity matrix is

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Can you see why multiplying this matrix by any 2×2 matrix results in the same 2×2 matrix?

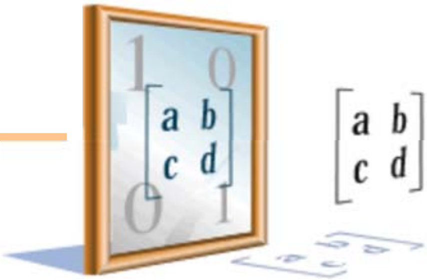
The identity matrix in Example A is the identity matrix for all 2×2 matrices. Take a minute to multiply $[I][A]$ and $[A][I]$ with any 2×2 matrix. There are corresponding identity matrices for larger square matrices.

Identity Matrix

An **identity matrix,** symbolized by $[I]$, is the square matrix that does not alter the entries of a square matrix $[A]$ under multiplication.

$$[A][I] = [A] \text{ and } [I][A] = [A]$$

Matrix $[I]$ must have the same dimensions as matrix $[A]$, and it has entries of 1's along the main diagonal (from top left to bottom right) and 0's in all other entries.



Now that you know the identity matrix for a 2×2 matrix, you can look for a way to find the inverse of a 2×2 matrix.

Inverse Matrix

The **inverse matrix** of $[A]$, symbolized by $[A]^{-1}$, is the matrix that will produce an identity matrix when multiplied by $[A]$.

$$[A][A]^{-1} = [I] \text{ and } [A]^{-1}[A] = [I]$$

Investigation
The Inverse Matrix

In this investigation you will learn ways to find the inverse of a 2×2 matrix.

Step 1 Use the definition of an inverse matrix to set up a matrix equation. Use these matrices and the 2×2 identity matrix for $[I]$.

$$[A] = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} \qquad [A]^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Step 2 Use matrix multiplication to find the product of $[A][A]^{-1}$. Set that product equal to matrix $[I]$.

Step 3 Use the matrix equation from Step 2 to write equations that you can solve to find values for a, b, c, and d. Solve the systems to find the values in the inverse matrix.

Step 4 Use your calculator to find $[A]^{-1}$. If this answer does not match your answer to Step 3, check your work for mistakes. [See Calculator Note 6F to learn how to find the inverse on your calculator.]

Step 5 Find the products of $[A][A]^{-1}$ and $[A]^{-1}[A]$. Do they both give you $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$? Is matrix multiplication always commutative?

Step 6 Not every square matrix has an inverse. Find the inverse of each of these matrices, if one exists. Make a conjecture about what types of 2×2 square matrices do not have inverses.

a. $\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$ **b.** $\begin{bmatrix} 50 & -75 \\ 10 & -15 \end{bmatrix}$ **c.** $\begin{bmatrix} 10.5 & 1 \\ 31.5 & 3 \end{bmatrix}$ **d.** $\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$

Step 7 Can a nonsquare matrix have an inverse? Why or why not?

You now know how to find the inverse of a square matrix, both by hand and on your calculator. You can use an inverse matrix to solve a system of equations.

Solving a System Using the Inverse Matrix

A system of equations in standard form can be written in matrix form as $[A][X] = [B]$, where $[A]$ is the coefficient matrix, $[X]$ is the variable matrix, and $[B]$ is the constant matrix. Multiplying both sides by the inverse matrix, $[A]^{-1}$, with the inverse on the left, gives the values of variables in matrix $[X]$, which is the solution to the system.

$[A][X] = [B]$	The system in matrix form.
$[A]^{-1}[A][X] = [A]^{-1}[B]$	Left-multiply both sides by the inverse.
$[I][X] = [A]^{-1}[B]$	By the definition of inverse, $[A]^{-1}[A] = [I]$.
$[X] = [A]^{-1}[B]$	By the definition of identity, $[I][X] = [X]$.

EXAMPLE B

Solve this system using an inverse matrix.

$$\begin{cases} 2x + 3y = 7 \\ x = 6 - 4y \end{cases}$$

▸ Solution

First, rewrite the second equation in standard form.

$$\begin{cases} 2x + 3y = 7 \\ x + 4y = 6 \end{cases}$$

The matrix equation for this system is

$$\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$$

In the matrix equation $\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$, the variable matrix, [*X*], is $\begin{bmatrix} x \\ y \end{bmatrix}$.

The coefficient matrix, [*A*], is $\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$, and the constant matrix, [*B*], is $\begin{bmatrix} 7 \\ 6 \end{bmatrix}$.

Use your calculator to find the inverse of [*A*].

$$[A]^{-1} = \begin{bmatrix} 0.8 & -0.6 \\ -0.2 & 0.4 \end{bmatrix}$$

Multiply both sides of the equation by this inverse, with the inverse on the left, to find the solution to the system of equations.

$\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$	$[A][X] = [B]$	The system in matrix form.
$\begin{bmatrix} 0.8 & -0.6 \\ -0.2 & 0.4 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0.8 & -0.6 \\ -0.2 & 0.4 \end{bmatrix}\begin{bmatrix} 7 \\ 6 \end{bmatrix}$	$[A]^{-1}[A][X] = [A]^{-1}[B]$	Left-multiply both sides by the inverse.
$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0.8 & -0.6 \\ -0.2 & 0.4 \end{bmatrix}\begin{bmatrix} 7 \\ 6 \end{bmatrix}$	$[I][X] = [A]^{-1}[B]$	By the definition of inverse, $[A]^{-1}[A] = [I]$.
$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0.8 & -0.6 \\ -0.2 & 0.4 \end{bmatrix}\begin{bmatrix} 7 \\ 6 \end{bmatrix}$	$[X] = [A]^{-1}[B]$	By the definition of identity, $[I][X] = [X]$.
$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$		Multiply the right side.

The solution to the system is (2, 1). Substitute the values into the original equations to check the solution.

$2x + 3y = 7$	$x = 6 - 4y$
$2(2) + 3(1) \stackrel{?}{=} 7$	$2 \stackrel{?}{=} 6 - 4(1)$
$4 + 3 \stackrel{?}{=} 7$	$2 \stackrel{?}{=} 6 - 4$
$7 = 7$	$2 = 2$

The solution checks.

You can also solve larger systems of equations using an inverse matrix. First, decide what quantities are unknown and write equations using the information given in the problem. Then rewrite the system of equations as a matrix equation, and use either row reduction or an inverse matrix to solve the system. Even systems of equations with many unknowns can be solved quickly this way.

EXAMPLE C

On a recent trip to the movies, Duane, Marsha, and Parker each purchased snacks. Duane bought two candy bars, a small drink, and two bags of popcorn for a total of $11.85. Marsha spent $9.00 on a candy bar, two small drinks, and a bag of popcorn. Parker spent $12.35 on two small drinks and three bags of popcorn, but no candy. If all the prices included tax, what was the price of each item?

Solution

The prices of the items are the unknowns. Let c represent the price of a candy bar in dollars, let d represent the price of a small drink in dollars, and let p represent the price of a bag of popcorn in dollars. This system represents the three friends' purchases:

$$\begin{cases} 2c + 1d + 2p = 11.85 \\ 1c + 2d + 1p = 9.00 \\ 0c + 2d + 3p = 12.35 \end{cases}$$

Translate these equations into a matrix equation in the form $[A][X] = [B]$.

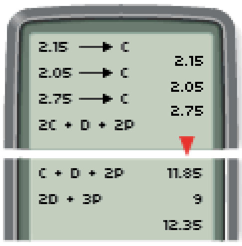
$$\begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 1 \\ 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} c \\ d \\ p \end{bmatrix} = \begin{bmatrix} 11.85 \\ 9.00 \\ 12.35 \end{bmatrix}$$

The solution to the system is simply the product $[A]^{-1}[B]$.

$$[X] = [A]^{-1}[B] = \begin{bmatrix} 2.15 \\ 2.05 \\ 2.75 \end{bmatrix}$$

A candy bar costs $2.15, a small drink costs $2.05, and a bag of popcorn costs $2.75.

Substituting these answers into the original system shows that they are correct. You can use your calculator to evaluate the expressions quickly and accurately.



You have probably noticed that in order to solve systems of equations with two variables, you must have two equations. To solve a system of equations with three variables, you must have three equations. In general, you must have as many equations as variables. Otherwise, there will not be enough information to solve the problem. For matrix equations, this means that the coefficient matrix must be square.

If there are more equations than variables, often one equation is equivalent to another equation and therefore just repeats the same information. Or the extra information may contradict the other equations, and thus there is no solution that will satisfy all the equations.

EXERCISES

Practice Your Skills

1. Rewrite each system of equations in matrix form.

a. $\begin{cases} 3x + 4y = 11 \\ 2x - 5y = -8 \end{cases}$

b. $\begin{cases} x + 2y + z = 0 \\ 3x - 4y + 5z = -11 \\ -2x - 8y - 3z = 1 \end{cases}$

c. $\begin{cases} 5.2x + 3.6y = 7 \\ -5.2x + 2y = 8.2 \end{cases}$

d. $\begin{cases} \frac{1}{4}x - \frac{2}{5}y = 3 \\ \frac{3}{8}x + \frac{2}{5}y = 2 \end{cases}$

2. Multiply each pair of matrices. If multiplication is not possible, explain why.

a. $\begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}\begin{bmatrix} 1 & -3 \\ 5 & -2 \end{bmatrix}$ **b.** $\begin{bmatrix} 4 & -1 \\ 3 & 6 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} 2 & -5 & 0 \\ 1 & -2 & 7 \end{bmatrix}$ **c.** $\begin{bmatrix} 9 & -3 \end{bmatrix}\begin{bmatrix} 4 & -6 \\ 0 & -2 \\ -1 & 3 \end{bmatrix}$

3. Use matrix multiplication to expand each system. Then solve for each variable by using substitution or elimination.

a. $\begin{bmatrix} 1 & 5 \\ 6 & 2 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}=\begin{bmatrix} -7 & 33 \\ 14 & -26 \end{bmatrix}$ **b.** $\begin{bmatrix} 1 & 5 \\ 6 & 2 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

4. Multiply each pair of matrices. Are the matrices inverses of each other?

a. $\begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}$ **b.** $\begin{bmatrix} 1 & 5 & 4 \\ 6 & 2 & -2 \\ 0 & 3 & 1 \end{bmatrix}\begin{bmatrix} 0.16 & 0.14 & -0.36 \\ -0.12 & 0.02 & 0.52 \\ 0.36 & -0.06 & -0.56 \end{bmatrix}$

5. Find the inverse of each matrix by solving the matrix equation $[A][A]^{-1} = [I]$. Then find the inverse matrix on your calculator to check your answer.

a. $\begin{bmatrix} 4 & 3 \\ 5 & 4 \end{bmatrix}$ **b.** $\begin{bmatrix} 6 & 4 & -2 \\ 3 & 1 & -1 \\ 0 & 0 & 3 \end{bmatrix}$ **c.** $\begin{bmatrix} 5 & 3 \\ 10 & 7 \end{bmatrix}$ **d.** $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

Reason and Apply

6. Rewrite each system in matrix form and solve by using the inverse matrix. Check your solutions.

a. $\begin{cases} 8x+3y=41 \\ 6x+5y=39 \end{cases}$ **b.** $\begin{cases} 11x-5y=-38 \\ 9x+2y=-25 \end{cases}$

c. $\begin{cases} 2x+y-2z=1 \\ 6x+2y-4z=3 \\ 4x-y+3z=5 \end{cases}$ **d.** $\begin{cases} 4w+x+2y-3z=-16 \\ -3w+3x-y+4z=20 \\ 5w+4x+3y-z=-10 \\ -w+2x+5y+z=-4 \end{cases}$

7. At the High Flying Amusement Park there are three kinds of rides: Jolly rides, Adventure rides, and Thrill rides. Admission is free when you buy a book of tickets, which includes ten tickets for each type of ride. Or you can pay \$5.00 for admission and then buy tickets for each of the rides individually. Noah, Rita, and Carey decide to pay the admission price and buy individual tickets. Noah pays \$19.55 for 7 Jolly rides, 3 Adventure rides, and 9 Thrill rides. Rita pays \$13.00 for 9 Jolly rides, 10 Adventure rides, and no Thrill rides. Carey pays \$24.95 for 8 Jolly rides, 7 Adventure rides, and 10 Thrill rides. (The prices above do not include the admission price.)

a. How much does each type of ride cost?

b. What is the total cost of a 30-ride book of tickets?

c. Would Noah, Rita, or Carey have been better off purchasing a ticket book?

8. A family invested a portion of \$5000 in an account at 6% annual interest and the rest in an account at 7.5% annual interest. The total interest they earned in the first year was \$340.50. How much did they invest in each account?

9. The midsize angle of a triangle is 30° greater than the smallest angle. The largest angle is 10° more than twice the midsize angle. What are the measures of the three angles?

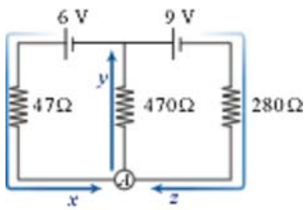
10. Being able to solve a system of equations is definitely not "new" mathematics. Mahāvīra, the best-known Indian mathematician of the 9th century, worked the following problem. See if you can solve it.

The mixed price of 9 citrons and 7 fragrant wood apples is 107; again, the mixed price of 7 citrons and 9 fragrant wood apples is 101. O you arithmetician, tell me quickly the price of a citron and of a wood apple here, having distinctly separated those prices well.

History CONNECTION

Mahāvīra (ca. 800-870 C.E.) wrote *Ganita Sara Samgraha,* the first Indian text exclusively about mathematics. Writing in his home of Mysore, India, he considered this book to be a collection of insights from other Indian mathematicians, such as Āryabata I, Bhāskara I, and Brahmagupta, who wrote their findings in astronomy texts.

11. **APPLICATION** The circuit here is made of two batteries (6 volt and 9 volt) and three resistors (47 ohms, 470 ohms, and 280 ohms). The batteries create an electric current in the circuit. Let *x*, *y*, and *z* represent the current in amps flowing through each resistor. The voltage across each resistor is current times resistance ($V = IR$). This gives two equations for the two loops of the circuit:



$47x + 470y = 6 \qquad 280z + 470y = 9$

The electric current flowing into any point along the circuit must flow out. So, for instance, at junction *A*, $x + z - y = 0$. Find the current flowing through each resistor.

12. When you use your calculator to find the inverse of the coefficient matrix for this system, you get an error message. What does this mean about the system?

$$\begin{cases} 3.2x + 2.4y = 9.6 \\ 2x + 1.5y = 6 \end{cases}$$

13. One way to find an inverse of a matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, if it exists, is to perform row operations on the augmented matrix $\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right]$ to change it to the form $\left[\begin{array}{cc|cc} 1 & 0 & e & f \\ 0 & 1 & g & h \end{array}\right]$. The matrix $\begin{bmatrix} e & f \\ g & h \end{bmatrix}$ is the required inverse. Use this strategy to find the inverse of each matrix.

a. $\begin{bmatrix} 4 & 3 \\ 5 & 4 \end{bmatrix}$

b. $\begin{bmatrix} 6 & 4 & -2 \\ 3 & 1 & -1 \\ 0 & 7 & 3 \end{bmatrix}$

14. **APPLICATION** An important application in the study of economics is the study of the relationship between industrial production and consumer demand. In creating an economic model, Russian-American economist Wassily Leontief (1906–1999) noted that the total output less the internal consumption equals consumer demand. Mathematically his input-output model looks like $[X] - [A][X] = [D]$, where $[X]$ is the total output matrix, $[A]$ is the input-output matrix, and $[D]$ is the matrix representing consumer demand.

Here is an input-output matrix, $[A]$, for a simple three-sector economy:

Input \ Output	Agriculture	Manufacturing	Service
Agriculture	0.2	0.2	0.1
Manufacturing	0.2	0.4	0.1
Servic	0.1	0.2	0.3

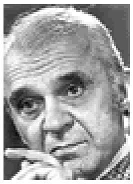
For instance, the first column tells the economist that to produce an output of 1 unit of agricultural products requires the consumption (input) of 0.2 unit of agricultural products, 0.2 unit of manufacturing products, and 0.1 unit of service products.

The demand matrix, $[D]$, represents millions of dollars. Use the equation $[X] - [A][X] = [D]$ to find the output matrix, $[X]$.

$$[D] = \begin{bmatrix} 100 \\ 80 \\ 50 \end{bmatrix}$$

Economics CONNECTION

During World War II, Wassily Leontief's method became a critical part of planning for wartime production in the United States. As a consultant to the U.S. Labor Department, he developed an input-output table for more than 90 economic sectors. During the early 1960s, Leontief and economist Marvin Hoffenberg used input-output analysis to forecast the economic effects of reduction or elimination of militaries. In 1973, Leontief was awarded a Nobel Prize in economics for his contributions to the field.



Wassily Leontief

Review

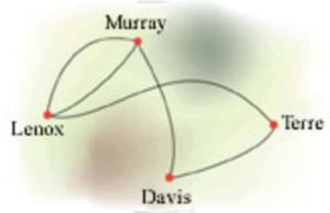
15. For each equation, write a second linear equation that would create a consistent and dependent system.

a. $y = 2x + 4$ **b.** $y = -\frac{1}{3}x - 3$ **c.** $2x + 5y = 10$ **d.** $x - 2y = -6$

16. For each equation, write a second linear equation that would create an inconsistent system.

a. $y = 2x + 4$ **b.** $y = -\frac{1}{3}x - 3$ **c.** $2x + 5y = 10$ **d.** $x - 2y = -6$

17. Four towns, Lenox, Murray, Davis, and Terre, are connected by a series of roads.



a. Represent the number of direct road connections between the towns in a matrix, [*A*]. List the towns in the order Lenox, Murray, Davis, and Terre.

b. Explain the meaning of the value of a_{22}.

c. Describe the symmetry of your matrix.

d. How many roads are there? What is the sum of the entries in the matrix? Explain the relationship between these two answers.

e. Assume any one of the roads is one-way. How does this change your matrix in 17a?

18. The third term of an arithmetic sequence is 28. The seventh term is 80.What is the first term?

IMPROVING YOUR REASONING SKILLS

Secret Survey

Eric is doing a survey. He has a deck of cards and two questions written on a sheet of paper. He says, "Pick a card from the deck. Don't show it to me. If it is a red card, answer Question 1. If it is a black card, answer Question 2."

Question 1 (red card): Does your phone number end in an even number?

Question 2 (black card): Do you own a stuffed animal?

You pick a card and look at the paper, and you respond, "Yes." Eric records your answer, shuffles the cards, and goes on to the next person.

At the end of the survey, Eric has gathered 37 yeses and 23 noes. He calculates that 73% responded "yes" to the second question.

Explain how Eric was able to find this result without knowing which question each person was answering.

LESSON 6.5

Systems of Linear Inequalities

Frequently, real-world situations involve a range of possible values. Algebraic statements of these situations are called **inequalities.**

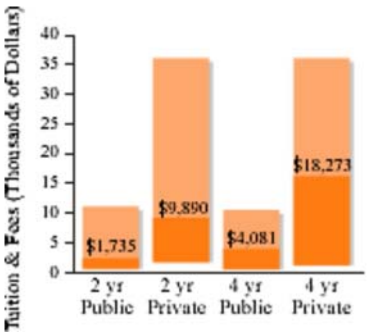
Situation	Inequality
Write an essay between two and five pages in length.	$2 \leq E \leq 5$
Practice more than an hour each day.	$P > 1$
The post office is open from nine o'clock until noon.	$9 \leq H \leq 12$
Do not spend more than $10 on candy and popcorn.	$c + p \leq 10$
A college fund has $40,000 to invest in stocks and bonds.	$s + b \leq 40000$

Recall that you can perform operations on inequalities very much like you do on equations. You can add or subtract the same quantity on both sides, multiply by the same number or expression on both sides, and so on. The one exception to remember is that when you multiply or divide by a negative quantity or expression, the inequality symbol reverses.

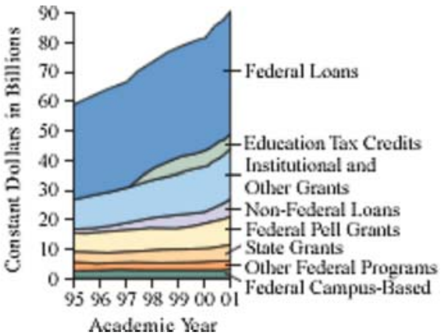
In this lesson you will learn how to graphically show solutions to inequalities with two variables, such as the last two statements in the table above.

The cost of a college education continues to rise. The good news is that over $90 billion is available in financial aid. At four-year public colleges, for example, over 60% of students receive some form of financial aid. Financial aid makes college affordable for many students, despite increasing costs.

Range and Weighted Mean of College Tuition and Fees, 2002-2003



Trends in College Financial Aid



Investigation
Paying for College

A total of $40,000 has been donated to a college scholarship fund. The administrators of the fund are considering how much to invest in stocks and how much to invest in bonds. Stocks usually pay more but are a riskier investment, whereas bonds pay less but are safer.

Step 1 Let x represent the amount in dollars invested in stocks, and let y represent the amount in dollars invested in bonds. Graph the equation $x + y = 40000$.

Step 2 Name at least five pairs of x- and y-values that satisfy the inequality $x + y < 40000$ and plot them on your graph. In this problem, why can $x + y$ be less than $40,000?

Step 3 Describe where all possible solutions to the inequality $x + y < 40000$ are located. Shade this region on your graph.

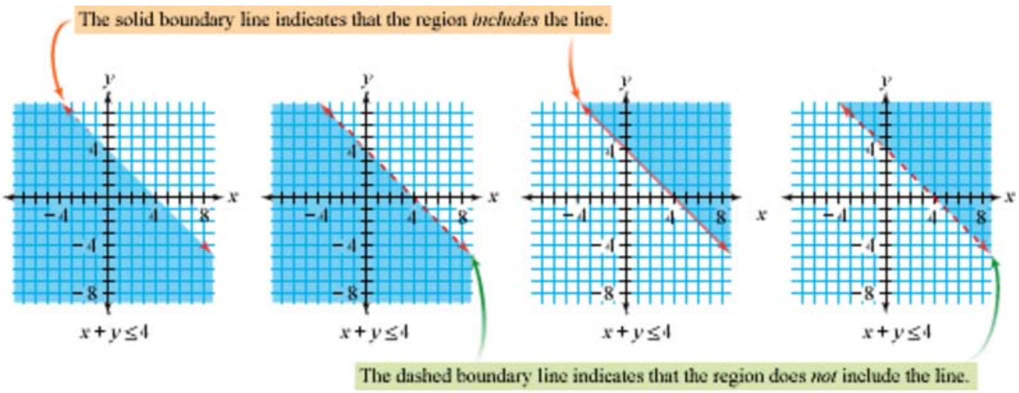
Step 4 Describe some points that fit the condition $x + y \leq 40000$ but do not make sense for the situation.

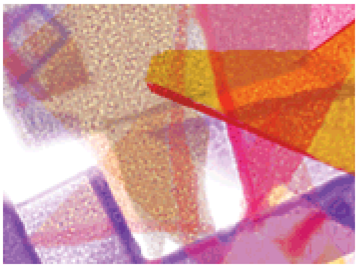
Assume that each option-stocks or bonds-requires a minimum investment of $5000, and that the fund administrators want to purchase some stocks and some bonds. Based on the advice of their financial advisor, they decide that the amount invested in bonds should be at least twice the amount invested in stocks.

Step 5 Translate all of the limitations, or **constraints,** into a system of inequalities. A table might help you to organize this information.

Step 6 Graph all of the inequalities and determine the region of your graph that will satisfy all the constraints. Find each corner, or **vertex,** of this region.

When there are one or two variables in an inequality, you can represent the solution as a set of ordered pairs by shading the region of the coordinate plane that contains those points.

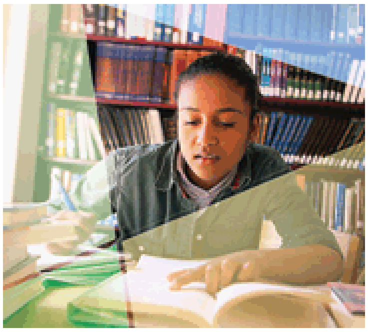




When you have several inequalities that must be satisfied simultaneously, you have a system. The solution to a system of inequalities with two variables will be a set of points rather than a single point. This set of points is called a **feasible region.** The feasible region can be shown graphically as part of a plane, or sometimes it can be described as a geometric shape with its vertices given.

EXAMPLE

Rachel has 3 hours to work on her homework tonight. She wants to spend more time working on mathematics than on chemistry, and she must spend at least a half hour working on chemistry. State the constraints of this system algebraically with x representing mathematics time in hours and y representing chemistry time in hours. Graph your inequalities, shade the feasible region, and label its vertices.

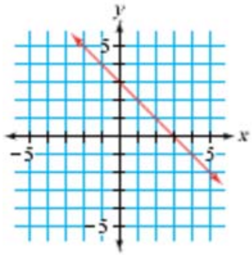


▸ Solution

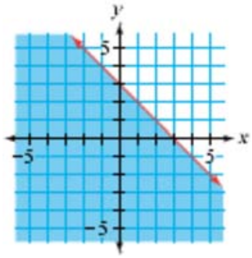
Convert each constraint into an algebraic inequality.

$$\begin{cases} x + y \le 3 & \text{Rachel has 3 h to work on homework.} \\ x > y & \text{She wants to spend more time working on mathematics than on chemistry.} \\ y \ge 0.5 & \text{She must spend at least a half hour working on chemistry.} \end{cases}$$

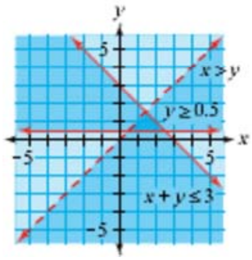
To graph the inequality $x + y \le 3$, or $y \le -x + 3$, first graph $y = -x + 3$. Use a solid line because y can be equal to $-x + 3$. This line divides the plane into two regions. One region contains all of the points for which y is less than $-x + 3$, and the other region contains all of the points for which y is greater than $-x + 3$.



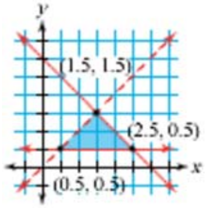
To find which side to shade, choose a sample point on either side of the line. Test the coordinates of this point in the inequality to see if it makes a true statement. If it does, then it falls in the region that satisfies the inequality, so shade on the side of the line where the point lies. If it doesn't, shade on the other side of the line. For example, if you choose (0, 0), $0 \le -0 + 3$, or $0 \le 3$, is a true statement, so shade on the side of the boundary line that contains (0, 0).



Follow the same steps to graph $x > y$ and $y \ge 0.5$ on the same axes. The graph of $x > y$ will contain a dashed line. [▸ See Calculator Note 6G to learn how to graph systems of inequalities using your calculator. ◂]



The solution to the system is the set of points in the area that represents the overlap of all the shaded regions-the feasible region. Every point in this region is a possible solution to the system.



To define the area that represents the solutions, name the vertices of the region. You can find the vertices by finding the intersection of each pair of equations using substitution, elimination, or matrices.

Equations	Intersections
$x + y = 3$ and $x = y$	(1.5, 1.5)
$x + y = 3$ and $y = 0.5$	(2.5, 0.5)
$x = y$ and $y = 0.5$	(0.5, 0.5)

The solution to this system is all points in the interior of a triangle with vertices (1.5, 1.5), (2.5, 0.5), and (0.5, 0.5), and the points on the lower and right edges of the triangle. Any point within this region represents a way that Rachel could divide her time. For example, (1.5, 1) means she could spend 1.5 h on mathematics and 1 h on chemistry and still meet all her constraints. Notice, however, that (0.5, 0.5) is not a solution to the system, even though it is a vertex of the feasible region. The point (0.5, 0.5) does not meet the constraint $x > y$.

When you are solving a system of equations based on real-world constraints, it is important to note that sometimes there are constraints that are not specifically stated in the problem. In the example, negative values for x and y would not make sense, because you can't study for a negative number of hours. You could have added the commonsense constraints $x \geq 0$ and $y \geq 0$, although in the example it would not affect the feasible region.

EXERCISES

Practice Your Skills

1. Solve each inequality for y.

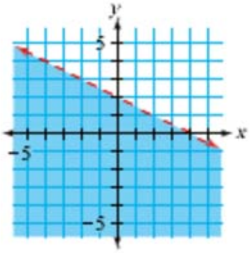
a. $2x - 5y > 10$ **b.** $4(2 - 3y) + 2x > 14$

2. Graph each linear inequality.

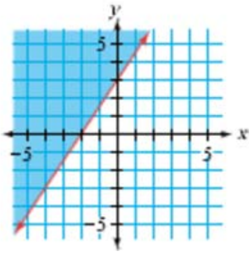
a. $y \leq -2x + 5$ **b.** $2y + 2x > 5$ **c.** $x > 5$

3. For 3a-d, write the equation of each graph.

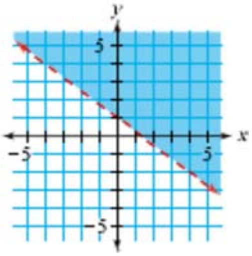
a.



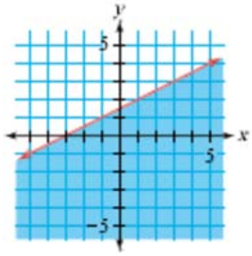
b.



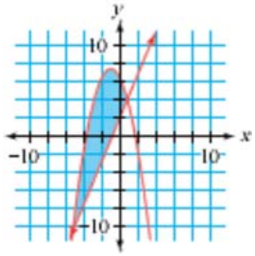
c.



d.



4. The graphs of $y = 2.4x + 2$ and $y = -x^2 - 2x + 6.4$ serve as the boundaries of this feasible region. What two inequalities identify this region?



Keymath.com More Practice Your Skills

Reason and Apply

For Exercises 5-8, sketch the feasible region of each system of inequalities. Find the coordinates of each vertex.

5. $\begin{cases} y \le -0.51x + 5 \\ y \le -1.6x + 8 \\ y \ge 0.1x + 2 \\ y \ge 0 \\ x \ge 0 \end{cases}$

6. $\begin{cases} y \ge 1.6x - 3 \\ y \le -(x-2)^2 + 4 \\ y \ge 1 - x \\ y \ge 0 \\ x \ge 0 \end{cases}$

7. $\begin{cases} 4x + 3y \le 12 \\ 1.6x + 2y \le 8 \\ 2x + y \ge 2 \\ y \ge 0 \\ x \ge 0 \end{cases}$

8. $\begin{cases} y \ge |x - 1| \\ y \le \sqrt{9 - x^2} \\ y \le 2.5 \\ y \ge 0 \end{cases}$

This color-stain painting by American artist Morris Louis (1912-1962) shows overlapping regions similar to the graph of a system of inequalities.
Floral (1959) by Morris Louis

9. In the Lux Art Gallery, rectangular paintings must havean area between 200 $in.^2$ and 300 $in.^2$ and a perimeterbetween 66 in. and 80 in.

a. Write four inequalities involving length and widththat represent these constraints.

b. Graph this system of inequalities to identify thefeasible region.

c. Will the gallery accept a painting that measures

i. 12.4 in. by 16.3 in.?

ii. 16 in. by 17.5 in.?

iii. 14.3 in. by 17.5 in.?

American artist Nina Bovasso (b 1965) carries one of herpaintings to an exhibition called Vir Heroicus Snoopicus. Would this piece, titled *Black and White Crowd,* be acceptedat the Lux Art Gallery in Exercise 9?

10. **APPLICATION** As the altitude of a spacecraft increases, an astronaut's weight decreases until a state of weightlessness is achieved. The weight of a 57 kg astronaut, *W*, at a given altitude in kilometers above Earth's sea level, *x*, is given by the formula

$$W = 57 \cdot \frac{6400^2}{(6400 + x)^2}$$

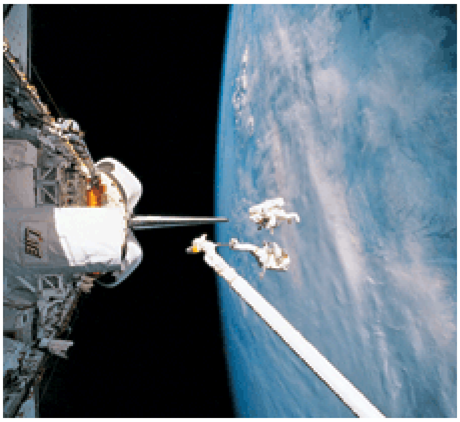
a. At what altitudes will the astronaut weigh less than 2 kg?

b. At an altitude of 400 km, how much will the astronaut weigh?

c. Will the astronaut ever be truly weightless? Why or why not?

Science CONNECTION

A typical space shuttle orbits at an altitude of about 400 km. At this height, an astronaut still weighs about 88.8% of her weight on Earth. You have probably seen pictures in which astronauts in orbit on a space shuttle or space station appear to be weightless. This is actually not due to the absence of gravity but, rather, to an effect called microgravity. In orbit, astronauts and their craft are being pulled toward Earth by gravity, but their speed is such that they are in free fall around Earth, rather than toward Earth. Because the astronauts and their spacecraft are falling through space at the same rate, the astronauts appear to be floating inside the craft. This is similar to the fact that a car's driver can appear to be sitting still, although he is actually traveling at a speed of 60 mi/h.



Astronauts floating in space during the 1994 testing of rescue system hardware appear to be weightless. One astronaut floats without being tethered to the spacecraft by using a small control unit.

11. Al just got rid of 40 of his dad's old records. He sold each classical record for \$5 and each jazz record for \$2. The rest of the records could not be sold, so he donated them to a thrift shop. Al knows that he sold fewer than 10 jazz records and that he earned more than \$100.

a. Let x represent the number of classical records sold, and let y represent the number of jazz records sold. Write an inequality expressing that Al earned more than \$100.

b. Write an inequality expressing that he sold fewer than 10 jazz records.

c. Write an inequality expressing that the total number of records sold was no more than 40.

d. Graph the solution to the system of inequalities, including any commonsense constraints.

e. Name all the vertices of the feasible region.

Review

12. A parabola with an equation in the form $y = ax^2 + bx + c$ passes through the points $(-2, -32)$, $(1, 7)$, and $(3, 63)$.

a. Set up systems and use matrices to find the values of a, b, and c for this parabola.

b. Write the equation of this parabola.

c. Describe how to verify that your answer is correct.

13. These data were collected from a bouncing-ball experiment. Recall that the height in centimeters, y, is exponentially related to the number of the bounce, x. Find the values of a and b for an exponential model in the form $y = ab^x$.

Bounce number	3	7
Height (cm)	34.3	8.2

14. Complete the reduction of this augmented matrix to row-echelon form. Give each row operation and find each missing matrix entry.

$$\left[\begin{array}{cc|c} 3 & -1 & 5 \\ -4 & 2 & 1 \end{array}\right] \xrightarrow{?} \left[\begin{array}{cc|c} 1 & ? & ? \\ -4 & 2 & 1 \end{array}\right] \xrightarrow{?} \left[\begin{array}{cc|c} 1 & ? & ? \\ 0 & ? & ? \end{array}\right] \xrightarrow{?} \left[\begin{array}{cc|c} 1 & ? & ? \\ 0 & 1 & ? \end{array}\right] \xrightarrow{?} \left[\begin{array}{cc|c} 1 & 0 & ? \\ 0 & 1 & ? \end{array}\right]$$

15. **APPLICATION** The growth of a population of water fungus is modeled by the function $y = f(x) = 2.68(3.84)^x$ where x represents the number of hours elapsed and y represents the number of fungi spores.

a. How many spores are there initially?
b. How many spores are there after 10 h?
c. Find an inverse function that uses y as the dependent variable and x as the independent variable.
d. Use your answer to 15c to find how long it will take until the number of spores exceeds 1 billion.

IMPROVING YOUR REASONING SKILLS

Coding and Decoding

One of the uses of matrices is in the mathematical field of cryptography, the science of enciphering and deciphering encoded messages. Here's one way that a matrix can be used to make a secret code:

Imagine encoding the word "CODE." First you convert each letter to a numerical value, based on its location in the alphabet. For example, C = 3 because it is the third letter in the alphabet. So, CODE = 3, 15, 4, 5. Arrange these numbers in a 2 × 2 matrix: $\begin{bmatrix} 3 & 15 \\ 4 & 5 \end{bmatrix}$.

Now multiply by an encoding matrix.

Let's use $[E] = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$, so $\begin{bmatrix} 3 & 15 \\ 4 & 5 \end{bmatrix}[E] = \begin{bmatrix} -12 & 27 \\ -1 & 6 \end{bmatrix}$.

Now you have to convert back to letters. Notice that three of these numbers are outside of the numbers 1-26 that represent A-Z. To convert other numbers, simply add or subtract 26 to make them fall within the range of 1 to 26, like this:

$$\begin{bmatrix} -12 & 27 \\ -1 & 6 \end{bmatrix} \longrightarrow \begin{bmatrix} 14 & 1 \\ 25 & 6 \end{bmatrix} \longrightarrow \begin{bmatrix} N & A \\ Y & F \end{bmatrix}$$

Now, to *decode* the encoded message "NAYF," you convert back to numbers and multiply by the inverse of the coding matrix, and then you convert each number back into the corresponding letter in the alphabet:

$$\begin{bmatrix} 14 & 1 \\ 25 & 6 \end{bmatrix}[E]^{-1} = \begin{bmatrix} 29 & 15 \\ 56 & 31 \end{bmatrix} = \begin{bmatrix} C & O \\ D & E \end{bmatrix}$$

See if you can decode this Navajo saying. Begin by taking out the spaces and breaking the letters into groups of four letters each.

CC FSLG GTQN YP OPIIY UCB DKIC BYF BEQQ WW URQLPRE.

Language CONNECTION

The Navajo language was used in coding messages by the U.S. Marines in World War II, because it is a complex unwritten language that is unintelligible to anyone without extensive training. Navajo code talkers could encode, transmit, and decode three-line English messages in 20 seconds, whereas machines of the time required 30 minutes to perform the same job. Navajo recruits developed the code, including a dictionary and numerous words for military terms. Learn more about Navajo code talkers by using the links at www.keymath.com/DAA.

Navajo recruits, like these two shown in the Pacific Island of Bougainville in 1943, shared their complex language with U.S. Marines to communicate in code during World War II.

LESSON 6.6

Keymath.com Links to Resources

Love the moment and the energy of that moment will spread beyond all boundaries.

CORITA KENT

Linear Programming

Industrial managers often investigate more economical ways of doing business. They must consider physical limitations, standards of quality, customer demand, availability of materials, and manufacturing expenses as restrictions, or constraints, that determine how much of an item they can produce. Then they determine the optimum, or best, amount of goods to produce-usually to minimize production costs or maximize profit. The process of finding a feasible region and determining the point that gives the maximum or minimum value to a specific expression is called **linear programming.**

Polish artist Roman Cieslewicz (1930-1996) formed this 1988 work by gluing strips of paper and fragments of photographs on cardboard.
Les dieux ont soif (1988) by Roman Cieslewicz

Problems that can be modeled with linear programming may involve anywhere from two variables to hundreds of variables. Computerized modeling programs that analyze up to 200 constraints and 400 variables are regularly used to help businesses choose their best plan of action. In this lesson you will look at problems that involve two variables because you are relying on the visual assistance of a two-dimensional graph to help you find the feasible region.

In this investigation you'll explore a linear programming problem and make conjectures about how to find the optimum value in the most efficient way.

Investigation
Maximizing Profit

The Elite Pottery Shoppe makes two kinds of birdbaths: a fancy glazed and a simple unglazed. An unglazed birdbath requires 0.5 h to make using a pottery wheel and 3 h in the kiln. A glazed birdbath takes 1 h on the wheel and 18 h in the kiln. The company's one pottery wheel is available for at most 8 hours per day (h/d). The three kilns can be used a total of at most 60 h/d. The company has a standing order for 6 unglazed birdbaths per day, so it must produce at least that many. The pottery shop's profit on each unglazed birdbath is $10, and the profit on each glazed birdbath is $40. How many of each kind of birdbath should the company produce each day in order to maximize profit?

Step 1 Organize the information into a table like this one:

	Amount per unglazed birdbath	Amount per glazed birdbath	Constraining value
Wheel hours			
Kiln hours			
Profit			Maximize

Step 2 Use your table to help you write inequalities that reflect the constraints given, and be sure to include any commonsense constraints. Let x represent the number of unglazed birdbaths, and let y represent the number of glazed birdbaths. Graph the feasible region to show the combinations of unglazed and glazed birdbaths the shop could produce, and label the coordinates of the vertices. (*Note:* Profit is not a constraint; it is what you are trying to maximize.)

Step 3 It will make sense to produce only whole numbers of birdbaths. List the coordinates of all integer points within the feasible region. (There should be 23.) Remember that the feasible region may include points on the boundary lines.

Step 4 Write the equation that will determine profit based on the number of unglazed and glazed birdbaths produced. Calculate the profit that the company would earn at each of the feasible points you found in Step 3. You may want to divide this task among the members of your group.

Step 5 What number of each kind of birdbath should the Elite Pottery Shoppe produce to maximize profit? What is the maximum profit possible? Plot this point on your feasible region graph. What do you notice about this point?

Step 6 Suppose that you want profit to be exactly \$100. What equation would express this? Carefully graph this line on your feasible region graph.

Step 7 Suppose that you want profit to be exactly \$140. What equation would express this? Carefully add this line to your graph.

Step 8 Suppose that you want profit to be exactly \$170. What equation would express this? Carefully add this line to your graph.

Step 9 How do your results from Steps 6-8 show you that (14, 1) must be the point that maximizes profit? Generalize your observations to describe a method that you can use with other problems to find the optimum value. What would you do if this vertex point did not have integer coordinates? What if you wanted to *minimize* profit?

Linear programming is a very useful real-world application of systems of inequalities. Its value is not limited to business settings, as the following example shows.

EXAMPLE

Marco is planning a snack of graham crackers and blueberry yogurt to provide at his school's track practice. Because he is concerned about health and nutrition, he wants to make sure that the snack contains no more than 700 calories and no more than 20 g of fat. He also wants at least 17 g of protein and at least 30% of the daily recommended value of iron. The nutritional content of each food is listed in the table below. Each serving of yogurt costs $0.30 and each graham cracker costs $0.06. What combination of servings of graham crackers and blueberry yogurt should Marco provide to minimize cost?

	Serving	Calories	Fat	Protein	Iron (percent of daily recommended value)
Graham crackers	1 cracker	60	2 g	2 g	6%
Blueberry yogurt	4.5 oz	130	2 g	5 g	1%

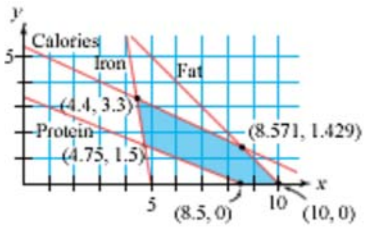
Solution

First organize the constraint information into a table, then write inequalities that reflect the constraints. Be sure to include any commonsense constraints. Let *x* represent the number of servings of graham crackers, and let *y* represent the number of servings of yogurt.

	Amount per graham cracker	Amount per serving of yogurt	Limiting value
Calories	60	130	≤ 700
Fat	2 g	2 g	≤ 20 g
Protein	2 g	5 g	≥ 17 g
Iron	6%	1%	≥ 30%
Cost	$0.06	$0.30	Minimize

$$\begin{cases} 60x + 130y \le 700 & \text{Calories} \\ 2x + 2y \le 20 & \text{Fat} \\ 2x + 5y \ge 17 & \text{Protein} \\ 6x + 1y \ge 30 & \text{Iron} \\ x \ge 0 & \text{Common sense} \\ y \ge 0 & \text{Common sense} \end{cases}$$

Now graph the feasible region and find the vertices.



Next, write an equation that will determine the cost of a snack based on the number of servings of graham crackers and yogurt.

Cost = 0.06*x* + 0.30*y*

You could try any possible combination of graham crackers and yogurt that is in the feasible region, but recall that in the investigation it appeared that optimum values will occur at vertices Calculate the cost at each of the vertices to see which one is a minimum.

The least expensive combination would be 8.5 crackers and no yogurt. However, Marco wants to serve only whole numbers of servings. The points (8, 1) and (9, 0) are the integer points within the feasible region closest to (8.5, 0), so test which point has a lower cost. The point (8, 1) gives a cost of $0.78 and (9, 0) will cost $0.54. Therefore Marco should serve 9 graham crackers and no yogurt.

x	*y*	Cost
4.444	3.333	$1.27
4.75	1.5	$0.74
8.571	1.429	$0.94
10	0	$0.60
8.5	0	$0.51

The following box summarizes the steps of solving a linear programming problem. Refer to these steps as you do the exercises.

Solving a Linear Programming Problem

1. Define your variables, and write constraints using the information given in the problem. Don't forget commonsense constraints.
2. Graph the feasible region, and find the coordinates of all vertices.
3. Write the equation of the function you want to optimize, and decide whether you need to maximize or minimize it.
4. Evaluate your optimization function at each of the vertices of your feasible region, and decide which vertex provides the optimum value.
5. If your possible solutions need to be limited to whole number values, and your optimum vertex does not contain integers, test the whole number values within the feasible region that are closest to this vertex.

EXERCISES

Practice Your Skills

1. Carefully graph this system of inequalities and label the vertices.

$$\begin{cases} x + y \le 10 \\ 5x + 2y \ge 20 \\ -x + 2y \ge 0 \end{cases}$$

2. For the system in Exercise 1, find the vertex that optimizes these expressions:

a. maximize: $5x + 2y$

b. minimize: $x + 3y$

c. maximize: $x + 4y$

d. minimize: $5x + y$

e. What generalizations can you make about which vertex provides a maximum or minimum value?

3. Graph this system of inequalities, label the vertices of the feasible region, and name the integer coordinates that maximize the function $P = 0.08x + 0.10y$. What is this maximum value of P?

$$\begin{cases} x \ge 5500 \\ y \ge 5000 \\ y \le 3x \\ x + y \le 40000 \end{cases}$$

4. **APPLICATION** During nesting season, two different bird species inhabit a region with area 180,000 m^2. Dr. Chan estimates that this ecological region can provide 72,000 kg of food during the season. Each nesting pair of species X needs 39.6 kg of food during a specified time period and 120 m^2 of land. Each nesting pair of species Y needs 69.6 kg of food and 90 m^2 of land. Let x represent the number of pairs of species X, and let y represent the number of pairs of species Y.

a. Describe the meaning of the constraints $x \geq 0$ and $y \geq 0$.

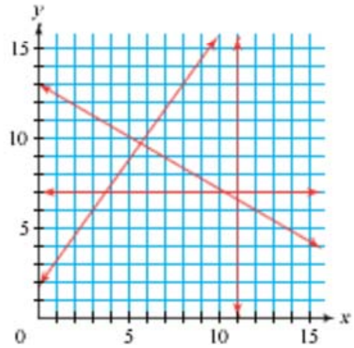
b. Describe the meaning of the constraint $120x + 90y \leq 180000$.

c. Describe the meaning of the constraint $39.6x + 69.6y \leq 72000$.

d. Graph the system of inequalities, and identify each vertex of the feasible region.

e. Maximize the total number of nesting pairs, N, by considering the function $N = x + y$.

Reason and Apply



5. Use a combination of the four lines shown on the graph along with the axes to create a system of inequalities whose graph satisfies each description.

a. The feasible region is a triangle.

b. The feasible region is a quadrilateral with one side on the y-axis.

c. The feasible region is a pentagon with sides on both the x-axis and the y-axis.

6. **APPLICATION** The International Canine Academy raises and trains Siberian sled dogs and dancing French poodles. Breeders can supply the academy with at most 20 poodles and 15 Siberian huskies each year. Each poodle eats 2 lb/d of food and each sled dog eats 6 lb/d. Food supplies are restricted to at most 100 lb/d. A poodle requires 1,000 h/yr of training, whereas a sled dog requires 250 h/yr. The academy cannot provide more than 15,000 h/yr of training time. If each poodle sells for a profit of $200 and each sled dog sells for a profit of $80, how many of each kind of dog should the academy raise in order to maximize profits?

7. **APPLICATION** The Elite Pottery Shoppe budgets a maximum of $1,000 per month for newspaper and radio advertising. The newspaper charges $50 per ad and requires at least four ads per month. The radio station charges $100 per minute and requires a minimum of 5 minutes of advertising per month. It is estimated that each newspaper ad reaches 8,000 people and that each minute of radio advertising reaches 15,000 people. What combination of newspaper and radio advertising should the business use in order to reach the maximum number of people? What assumptions did you make in solving this problem? How realistic do you think they are?

8. **APPLICATION** A small electric generating plant must decide how much low-sulfur (2%) and high-sulfur (6%) oil to buy. The final mixture must have a sulfur content of no more than 4%. At least 1200 barrels of oil are needed. Low-sulfur oil costs $18.50 per barrel and high-sulfur oil costs $14.70 per barrel. How much of each type of oil should the plant use to keep the cost at a minimum? What is the minimum cost?

9. **APPLICATION** A fair-trade farmers' cooperative in Chiapas, Mexico, is deciding how much coffee and cocoa to recommend to their members to plant. Their 1,000 member families have 7,500 total acres to farm. Because of the geography of the region, 2,450 acres are suitable only for growing coffee and 1,230 acres are suitable only for growing cocoa. A coffee crop produces 30 lb/acre and a crop of cocoa produces 40 lb/acre. The cooperative has the resources to ship a total of 270,000 lb of product to the United States. Fair-trade organizations mandate a minimum price of $1.26 per pound for organic coffee and $0.98 per pound for organic cocoa (note that price is per pound, not per acre). How many acres of each crop should the cooperative recommend planting in order to maximize income?


These coffee beans are growing in the Mexican state of Chiapas.

Consumer CONNECTION

Many small coffee and cocoa farmers receive prices for their crop that are less than the costs of production, causing them to live in poverty and debt. Fair-trade certification has been developed to show consumers which products are produced with the welfare of farming communities in mind. To become fair-trade certified, an importer must meet stringent international criteria: paying a minimum price per pound, providing credit to farmers, and providing technical assistance in farming upgrades. Fair-trade prices allow farmers to make enough money to provide their families with food, education, and health care.

10. **APPLICATION** Teo sells a set of videotapes on an online auction. The postal service he prefers puts these restrictions on the size of a package:

Up to 150 lb

Up to 130 in. in length and girth combined

Up to 108 in. in length

Length is defined as the longest side of a package or object. Girth is the distance all the way around the package or object at its widest point perpendicular to the length. Teo is not concerned with the weight because the videotapes weigh only 15 lb.

a. Write a system of inequalities that represents the constraints on the package size.
b. Graph the feasible region for the dimensions of the package.
c. Teo packages the videotapes in a box whose dimensions are 20 in. by 14 in. by 8 in. Does this box satisfy the restrictions?

Consumer CONNECTION

In the packaging industry, two sets of dimensions are used. Inside dimensions are used to ensure proper fit around a product to prevent damage. Outside dimensions are used in shipping classifications and determining how to stack boxes on pallets. In addition, the type of packaging material and its strength are important concerns. Corrugated cardboard is a particularly strong, yet economical, packaging material.

Review

11. Solve each of these systems in at least two different ways.

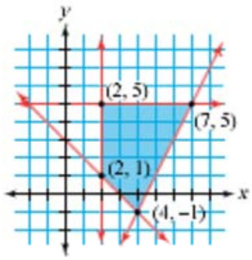
a. $\begin{cases} 8x + 3y = 41 \\ 9x + 2y = 25 \end{cases}$

b. $\begin{cases} 2x + y - 2z = 5 \\ 6x + 2y - 4z = 3 \\ 4x - y + 3z = 5 \end{cases}$

12. Sketch a graph of the feasible region described by this system of inequalities:

$$\begin{cases} y \geq (x-3)^2 + 5 \\ y \leq -|x-2| + 10 \end{cases}$$

13. Give the system of inequalities whose solution set is the polygon at right.



14. Consider this system of equations:

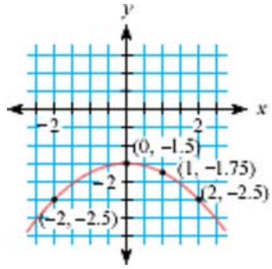
$$\begin{cases} y = 2x - \frac{1}{2} \\ 5x - 2y + 5 = 0 \end{cases}$$

a. Write the augmented matrix for this system.

b. Reduce the augmented matrix to row-echelon form. Write the solution as an ordered pair of coordinates.

c. Check your solution values for x and y by substituting them into the original equations.

15. Find the equation of the parabola at right.

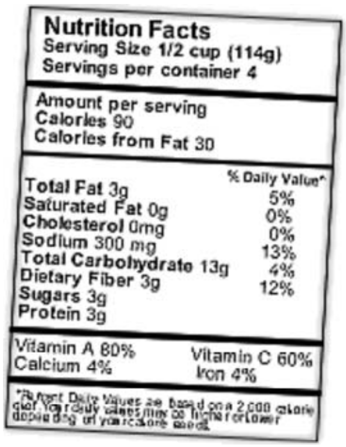


Project

NUTRITIONAL ELEMENTS

Write and solve your own linear programming problem similar to the cracker-and-yogurt snack combination problem in this lesson. Choose two food items from your home or a grocery store, and decide on your constraints and what you wish to minimize or maximize. Record the necessary information. Then write inequalities, and find the feasible region and optimum value for your problem. Your project should include

- Your linear programming problem
- A complete solution, including a graph of the feasible region
- An explanation of your process



CHAPTER 6 REVIEW

Keymath.com Links to Resources

Matrices have a variety of uses. They provide ways to organize data about such things as inventory or the coordinates of vertices of a polygon. A **transition matrix** represents repeated changes that happen to a system. You can use matrix arithmetic to combine data or transform polygons on a coordinate graph, and you can also use **matrix multiplication** to determine quantities at various stages of a transition simulation.

Another important application of matrices is to solve systems of equations. In Chapter 3, you solved systems of linear equations by looking for a point of intersection on a graph or by using substitution or elimination. With two linear equations, the system will be **consistent and independent** (intersecting lines with one solution), or **consistent and dependent** (the same line with infinitely many solutions), or **inconsistent** (parallel lines with no solution). You can use an **inverse matrix** and matrix multiplication to solve a system, or you can use an **augmented matrix** and the **row reduction** method. Matrix methods are generally the simplest way to solve systems that involve more than three equations and three variables.

When a system is made up of inequalities, the solution usually consists of many points that can be represented by a region in the plane. One important use of systems of inequalities is in **linear programming.** In linear programming problems, an equation for a quantity that is to be optimized (maximized or minimized) is evaluated at the vertices of the **feasible region.**

EXERCISES

1. Use these matrices to do the arithmetic problems 1a-d. If a particular operation is impossible, explain why.

$[A] = \begin{bmatrix} 1 & -2 \end{bmatrix}$ $\quad [B] = \begin{bmatrix} -3 & 7 \\ 6 & 4 \end{bmatrix}$ $\quad [C] = \begin{bmatrix} 1 & 0 \\ 5 & 2 \end{bmatrix}$ $\quad [D] = \begin{bmatrix} -3 & 1 & 2 \\ 2 & 3 & -2 \end{bmatrix}$

a. $[A] + [B]$ **b.** $[B] - [C]$ **c.** $4 \cdot [D]$ **d.** $[C]\,[D]$ **e.** $[D]\,[C]$ **f.** $[A]\,[D]$

2. Find the inverse, if it exists, of each matrix.

a. $\begin{bmatrix} 2 & -3 \\ 1 & -4 \end{bmatrix}$ **b.** $\begin{bmatrix} 5 & 2 & -2 \\ 6 & 1 & 0 \\ -2 & 5 & 3 \end{bmatrix}$ **c.** $\begin{bmatrix} -2 & 3 \\ 8 & -12 \end{bmatrix}$ **d.** $\begin{bmatrix} 5 & 2 & -3 \\ 4 & 3 & -1 \\ 7 & -2 & -1 \end{bmatrix}$

3. Solve each system by using row reduction.

a. $\begin{cases} 8x - 5y = -15 \\ 6x + 4y = 43 \end{cases}$ **b.** $\begin{cases} 5x + 3y - 7z = 3 \\ 10x - 4y + 6z = 5 \\ 15x + y - 8z = -2 \end{cases}$

4. Solve each system by using an inverse matrix.

a. $\begin{cases} 8x - 5y = -15 \\ 6x + 4y = 43 \end{cases}$

b. $\begin{cases} 5x + 3y - 7z = 3 \\ 10x - 4y + 6z = 5 \\ 15x + y - 8z = -2 \end{cases}$

5. Identify each system as consistent and independent (has one solution), inconsistent (has no solution), or consistent and dependent (has infinitely many solutions).

a. $\begin{cases} y = -1.5x + 7 \\ y = -3x + 14 \end{cases}$

b. $\begin{cases} y = \frac{1}{4}(x - 8) + 5 \\ y = 0.25x + 3 \end{cases}$

c. $\begin{cases} 2x + 3y = 4 \\ 1.2x + 1.8y = 2.6 \end{cases}$

d. $\begin{cases} \frac{3}{5}x - \frac{2}{5}y = 3 \\ 0.6x - 0.4y = -3 \end{cases}$

6. Graph the feasible region of each system of inequalities. Find the coordinates of each vertex. Then identify the point that maximizes the given expression.

a. $\begin{cases} 2x + 3y \leq 12 \\ 6x + y \leq 18 \\ x + 2y \geq 4 \\ x \geq 0 \\ y \geq 0 \end{cases}$

maximize: $1.65x + 5.2y$

b. $\begin{cases} x + y \leq 50 \\ 10x + 5y \leq 440 \\ 40x + 60y \leq 2400 \\ x \geq 0 \\ y \geq 0 \end{cases}$

maximize: $6x + 7y$

American artist Joseph Cornell (1903-1972) assembled sculptures from collections of objects, frequently drugstore items. This piece places medicine bottles in a matrix-like arrangement.

Untitled (Grand Hotel Pharmacy) (1947), Joseph Cornell

7. **APPLICATION** Heather's water heater needs repair. The plumber says it will cost $300 to fix the unit, which currently costs $75 per year to operate. Or Heather could buy a new energy-saving water heater for $500, including installation, and the new heater would save 60% on annual operating costs. How long would it take for the new unit to pay for itself?

8. **APPLICATION** A particular color of paint requires a mix of five parts red, six parts yellow, and two parts black. Thomas does not have the pure colors available, but he finds three pre-mixed colors that he can use. The first is two parts red and four parts yellow; the second is one part red and two parts black; the third is three parts red, one part yellow, and one part black.

a. Write an equation that gives the correct portion of red by using the three available pre-mixed colors.

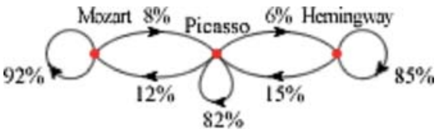
b. Write an equation that gives the correct portion of yellow and another equation that gives the correct portion of black.

c. Solve the systems of equations in 8a and b.

d. Find an integer that you can use as a scalar multiplier for your solutions in 8c to provide integer solution values.

e. Explain the real-world meaning to your solutions to 8d.

9. Interlochen Arts Academy 9th and 10th graders are housed in three dormitories: Picasso, Hemingway, and Mozart. Mozart is an all-female dorm, Hemingway is an all-male dorm, and Picasso is coed. In September, school started with 80 students in Mozart, 60 in Picasso, and 70 in Hemingway. Students are permitted to move from one dorm to another on the first Sunday of each month. This transition graph shows the movements this past year.



a. Write a transition matrix for this situation. List the dorms in the order Mozart, Picasso, Hemingway.

b. What were the populations of the dorms in

i. October?

ii. November?

iii. May?

10. **APPLICATION** Yolanda, Myriam, and Xavier have a small business producing handmade shawls and blankets. They spin the yarn, dye it, and weave it. A shawl requires 1 h of spinning, 1 h of dyeing, and 1 h of weaving. A blanket needs 2 h of spinning, 1 h of dyeing, and 4 h of weaving. They make a $16 profit per shawl and a $20 profit per blanket. Xavier does the spinning on his day off, when he can spend at most 8 h spinning. Yolanda dyes the yarn on her day off, when she has at most 6 h. Myriam does all the weaving on Friday and Saturday, when she has at most 14 h available. How many of each item should they make each week to maximize their profit?

MIXED REVIEW

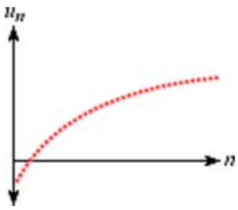
11. Graphs 11a-c were produced by a recursive formula in the form

$u_0 = a$

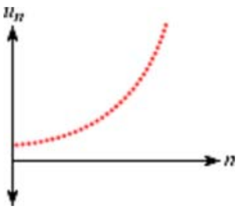
$u_n = u_{n-1}(1 + p) + d$ where $n \geq 1$

For each case, tell if *a*, *p*, and *d* are greater than zero (positive), equal to zero, or less than zero (negative).

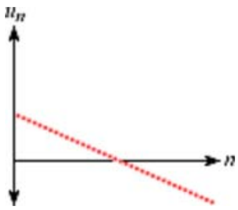
a.



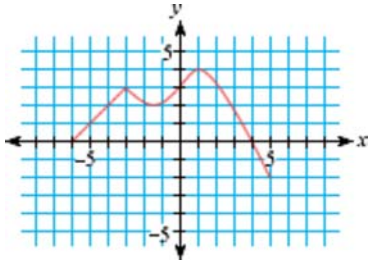
b.



c.



12. The graph of $y = f(x)$ is shown at right.



a. Find $f(-3)$.

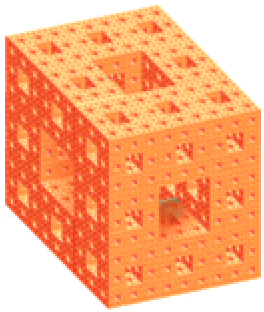
b. Find x such that $f(x) = 1$.

c. How can you use the graph to tell whether or not f is a function?

d. What is the domain of f?

e. What is the range of f?

13. Last semester, all of Ms. Nolte's students did projects. One-half of the students in her second-period class investigated fractals, one-fourth of the students in that class did research projects, and the remaining students conducted surveys and analyzed their results. In her third-period class, one-third of the students investigated fractals, one-half of the students did research projects, and the remaining students conducted surveys and analyzed their results. In the seventh-period class, one-fourth of the students investigated fractals, one-sixth of the students did research projects, and the remaining students conducted a survey and analyzed their results. Overall, 22 students investigated fractals, 18 students did research projects, and 22 students conducted surveys. How many students are in each of Ms. Nolte's classes?



14. **APPLICATION** Canada's oil production has increased over the last half century. This table gives the production of oil per day for various years.

Canada's Oil Production

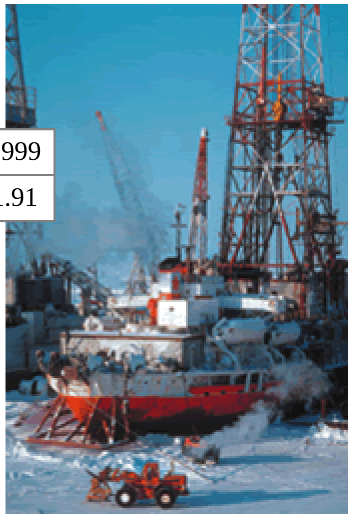
Year	1960	1970	1980	1990	1995	1998	1999
Barrels per day (millions)	0.52	1.26	1.44	1.55	1.80	1.98	1.91

(The New York Times Almanac 2002)

a. Define variables and make a scatter plot of the data.

b. Find M_1, M_2, and M_3, and write the equation of the median-median line.

c. Use the median-median line to predict Canada's oil production in 2002.



This oil-drilling ship was frozen in six feet of ice on the Beaufort Sea, Northwest Territories, Canada, in 1980.

15. Solve.

a. $\log 35 + \log 7 = \log x$

b. $\log 500 - \log 25 = \log x$

c. $\log \sqrt{\frac{1}{8}} = x \log 8$

d. $15(9.4)^x = 37000$

e. $\sqrt[3]{(x+6)} + 18.6 = 21.6$

f. $\log_6 342 = 2x$

16. **APPLICATION** Suppose the signal strength in a fiber-optic cable diminishes by 15% every 10 mi.

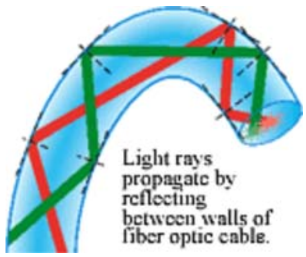
a. What percentage of the original signal strength is left after a stretch of 10 mi?

b. Create a table of the percentage of signal strength remaining in 10 mi intervals, and make a graph of the sequence.

c. If a phone company plans to boost the signal just before it falls to 1%, how far apart should the company place its booster stations?

Technology CONNECTION

Fiber-optic technology uses light pulses to transmit information from one transmitter to another down fiber lines made of silica (glass). Fiber-optic strands are used in telephone wires, cable television lines, power systems, and other communications. These strands operate on the principle of total internal reflection, which means that the light pulses cannot escape out of the glass tube and instead bounce information from transmitter to transmitter.



The photo on the left shows strands of fiber-optic cable. The illustration on the right shows how light is reflected along a strand of fiber-optic cable, creating total internal reflection.

17. The graph of an exponential function passes through the points (4, 50) and (6, 25.92).

a. Find the equation of the exponential function.

b. What is the rate of change? Does the equation model growth or decay?

c. What is the y-intercept?

d. What is the long-run value?

18. This data set gives the weights in kilograms of the crew members participating in the 2002 Boat Race between Oxford University and Cambridge University.
(www.cubc.org.uk and *www.ourcs.org)*

{83, 95, 91, 90, 93, 97.5, 97, 79, 55,
89, 89.5, 94, 89, 100, 90, 92, 96, 54}

a. Make a histogram of these data.

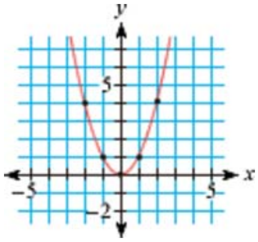
b. Are the data skewed left, skewed right, or symmetric?

c. Identify any outliers.

d. What is the percentile rank of the crew member who weighs 94 kg?

Photographed on London's Thames River in 1949, this television crew films the Oxford-Cambridge Boat Race, which was the British Broadcasting Corporation's first broadcast with equipment in motion.

19. Consider the graph of $y = x^2$. Let matrix $[P]$, which organizes the coordinates of the points shown, represent five points on the parabola.



$$[P] = \begin{bmatrix} -2 & -1 & 0 & 1 & 2 \\ 4 & 1 & 0 & 1 & 4 \end{bmatrix}$$

a. Describe the transformation(s) that would give five corresponding points on the graph of $y = (x - 5)^2 - 2$. Sketch the graph.

b. Describe the transformation(s) that would give five points on the graph of $y = -2x^2$. Sketch the graph.

c. Sketch the image of the portion of the graph of $y = x^2$ represented by the product $-1 \cdot [P]$. Describe the transformation(s).

d. Write a matrix equation that represents the image of five points on the graph of $y = x^2$ after a translation left 2 units and up 3 units.

20. **APPLICATION** This table gives the mean population per U.S. household for various years from 1890 to 1998.

Mean Household Population

Year	1890	1930	1940	1950	1960	1970	1980	1990	1998
Mean population	4.93	4.11	3.67	3.37	3.35	3.14	2.76	2.63	2.62

(*The New York Times Almanac 2001*)

a. Define variables and find a linear equation that fits these data.

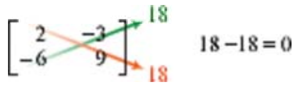
b. Write and solve a problem that you could solve by interpolation with your line of fit.

c. Write and solve a problem that you could solve by extrapolation with your line of fit.

A family poses before their sod house in Nebraska in 1887. Made from strips of earth and plant roots collected with a plow, sod was plastered with clay and ashes in a brick-like pattern. Sod houses were common in the Western U.S. plains because timber was scarce.

TAKE ANOTHER LOOK

1. You have probably noticed that when a matrix has no inverse, one of the rows is a multiple of another row. For a 2×2 matrix, this also means that the products of the diagonals are equal, or that the difference of these products is 0.



This difference of the diagonals is called the **determinant** of the matrix. For any 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the determinant is $ad - bc$.

Make up some 2×2 matrices that have a determinant with value 1. Find the inverses of these matrices. Describe the relationship between the entries of each matrix and its inverse matrix.

Make up some 2×2 matrices that have a determinant with value 2. Find the inverses of these matrices. Describe the relationship between the entries of each matrix and its inverse matrix.

Write a conjecture about the inverse of a matrix and how it relates to the determinant. Test your conjecture with several other 2×2 matrices. Does your conjecture hold true regardless of the value of the determinant?

2. In Lesson 6.3, Exercise 14, you multiplied $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ by a matrix that represented a triangle. The result was a rotation. Try multiplying this matrix by matrices representing other types of polygons. Does it have the same effect? Find a different rotation matrix-one that will rotate a polygon by a different amount or in a different direction. Show that it works for at least three different polygons. Explain why your matrix works.

3. You have learned how to do linear programming problems, but how would you do *nonlinear* programming? Carefully graph this system of inequalities and label all the vertices. Then find the point within the feasible region that maximizes the value of P in the equation $P = (x + 2)^2 + (y - 2)^2$. Explain your solution method and how you know that your answer is correct.

$$\begin{cases} y \le -|x+2| + 10 \\ 10^{y+2} \ge x + 8 \\ 3x + 8y \le 50 \\ -3y + x^2 \ge 9 \end{cases}$$

4. In Lesson 6.4, Exercise 15, you learned how to find the inverse of a square matrix using an augmented matrix. Use this method to find the inverse of the general 2×2 matrix, $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Compare your results to your work on Take Another Look 1, where you found the inverse using the determinant. Did you get the same results?

Assessing What You've Learned

PERFORMANCE ASSESSMENT While a classmate, a friend, a family member, or a teacher observes, show how to solve a system of equations using both an inverse matrix and the row reduction method. Explain all of your steps and why each method works.

WRITE IN YOUR JOURNAL Choose one or more of the following questions to answer in your journal.

- What kinds of matrices can be added to or subtracted from one another? What kinds of matrices can you multiply? Is matrix multiplication commutative? Why or why not? What are the identity matrix and the inverse matrix, and what are they used for?
- You have learned five methods to find a solution to a system of linear equations: graphing, substitution, elimination, matrix row reduction, and multiplication by an inverse matrix. Which method do you prefer? Which one is the most challenging to you? What are the advantages and disadvantages of each method?
- You have now studied half the chapters of this book. What mathematical skills in the previous chapters were most crucial to your success in this chapter? Which concepts are your strengths and weaknesses?

UPDATE YOUR PORTFOLIO Pick a linear programming problem for which you are especially proud of your work, and add it to your portfolio. Describe the steps you followed and how your graph helped you to solve the problem.